# Tikhonov Regularization for Stochastic Convex Optimization in Hilbert Spaces

Rodrigo Maulen-Soto* Jalal Fadili† Hedy Attouch‡

**Abstract.** To solve convex optimization problems with a noisy gradient input, we analyze the global behavior of subgradient-like flows under stochastic errors. The objective function is composite, being equal to the sum of two convex functions, one being differentiable and the other potentially non-smooth. We then use stochastic differential inclusions where the drift term is minus the subgradient of the objective function, and the diffusion term is either bounded or square-integrable. In this context, under Lipschitz's continuity of the differentiable term and a growth condition of the non-smooth term, our first main result shows almost sure weak convergence of the trajectory process towards a minimizer of the objective function. Then, using Tikhonov regularization with a properly tuned vanishing parameter, we can obtain almost sure strong convergence of the trajectory towards the minimum norm solution. We find an explicit tuning of this parameter when our objective function satisfies a local error-bound inequality. We also provide a comprehensive complexity analysis by establishing several new pointwise and ergodic convergence rates in expectation for the convex and strongly convex case.

**Key words.** Stochastic optimization, inertial gradient system, Convex optimization, Non-smooth optimization, Stochastic Differential Equation, Stochastic Differential Inclusion, Tikhonov regularization, Error bound inequality, Łojasiewicz inequality, KL inequality, Convergence rate, Asymptotic behavior.

**AMS subject classifications.** 37N40, 46N10, 49M99, 65B99, 65K05, 65K10, 90B50, 90C25, 60H10, 49J52.

# 1 Introduction

## 1.1 Problem statement

We aim to solve convex minimization problems by means of stochastic differential inclusions (SDI), showing the existence, uniqueness, and properties of the solution. Then, we work with Tikhonov regularization, specifically when the drift term is the sum of the (sub-)gradient of the objective function and of a Tikhonov regularization term with a vanishing coefficient. This makes it possible to take into account a noisy (imprecise) gradient input and obtain convergence a.s. to the minimal norm solution.

Let us consider the minimization problem

$$\min_{x\in\mathbb{H}} F(x) \stackrel{\text{def}}{=} f(x) + g(x), \tag{P}$$

*Normandie Université, ENSICAEN, UNICAEN, CNRS, GREYC, France. E-mail: rodrigo.maulen@ensicaen.fr
†Normandie Université, ENSICAEN, UNICAEN, CNRS, GREYC, France. E-mail: Jalal.Fadili@ensicaen.fr
‡IMAG, CNRS, Université Montpellier, France. E-mail: hedy.attouch@umontpellier.fr

where $\mathbb{H}$ is a separable real Hilbert space, and the objective $F$ satisfies the following standing assumptions:

$$\begin{cases} f : \mathbb{H} \to \mathbb{R} \text{ is continuously differentiable and convex with } L\text{-Lipschitz continuous gradient;} \\ g : \mathbb{H} \to \mathbb{R} \text{ is proper, lsc and convex;} \\ \mathcal{S}_F \stackrel{\text{def}}{=} \operatorname{argmin}(F) \neq \emptyset. \end{cases} \tag{H$_0$}$$

To solve (P), a fundamental dynamic to consider is the subgradient flow, which is the following differential inclusion (DI) starting in $t_0 \geq 0$ with initial condition $x_0 \in \mathbb{H}$:

$$\begin{cases} \dot{x}(t) \in -\partial F(x(t)), \quad t > t_0; \\ x(t_0) = x_0. \end{cases} \tag{DI}$$

It is well known since the founding articles of Brezis, Baillon, Bruck in the 1970s that, when the initial data $x_0$ is in the domain of $F$, (more generally when it is in its closure), there exists a unique strong global solution of (DI). Moreover, if the solution set $\operatorname{argmin}(F)$ of (P) is nonempty then each solution trajectory of (DI) converges weakly, and its limit belongs to $\operatorname{argmin}(F)$.

In many cases, the gradient input is subject to noise, for example, if the gradient cannot be evaluated directly, or due to some other exogenous factor. In such scenario, one can model the associated errors using a stochastic integral with respect to the measure defined by a continuous Itô martingale. This entails the following stochastic differential inclusion (SDI) as a stochastic counterpart of (DI), consider $\mathbb{K}$ a separable real Hilbert space:

$$\begin{cases} dX(t) \in -\partial F(X(t)) + \sigma(t, X(t))dW(t), \quad t \geq t_0; \\ X(t_0) = X_0, \end{cases} \tag{SDI}$$

where the diffusion (volatility) term $\sigma : [t_0, +\infty[\times \mathbb{H} \to \mathcal{L}_2(\mathbb{K}; \mathbb{H})$ (see notation in Section 2) is a measurable function, and $W$ is a $\mathbb{K}$-valued Brownian motion (see Section A.2.1 for a precise definition), and the initial data $X_0$ is an $\mathcal{F}_0$-measurable $\mathbb{H}$-valued random variable. This dynamic can be viewed as a stochastic dissipative system that aims to minimize $F$ if the diffusion term vanishes sufficiently fast. Also, it is the natural extension to the non-smooth setting of the work done in [1].

An important aspect of our work concerns the Tikhonov regularization of (DI) and (SDI). Given $t_0 > 0$, and a regularization parameter $\varepsilon : [t_0, +\infty[\to \mathbb{R}_+$, which is a measurable function that vanishes asymptotically in a controlled way, the Tikhonov regularization of (DI) is written:

$$\begin{cases} \dot{x}(t) \in -\partial F(x(t)) - \varepsilon(t)x(t), \quad t > t_0; \\ x(t_0) = x_0. \end{cases} \tag{DI-TA}$$

The stochastic counterpart of (DI-TA) (which is the Tikhonov regularization of (SDI)), is the following stochastic differential inclusion with initial data $X_0 \in \mathrm{L}^\nu(\Omega; \mathbb{H})$ (for some $\nu \geq 2$):

$$\begin{cases} dX(t) \in -\partial F(X(t)) - \varepsilon(t)X(t) + \sigma(t, X(t))dW(t), \quad t > t_0; \\ X(t_0) = X_0. \end{cases} \tag{SDI $-$ TA}$$

The impact of the Tikhonov term has been studied in depth in the deterministic case (DI-TA) (see [2]). The fact that the Tikhonov regularization parameter $\varepsilon(t)$ tends to zero not too fast as $t \to +\infty$ induces a hierarchical minimization property: the limit of any trajectory no longer depends on the initial data, it is precisely the minimum norm solution. We propose to extend these results to the stochastic case (SDI − TA) based on the recent work of Maulen-Soto, Fadili, and Attouch [1].

Our objective is to study the dynamics (SDI) and (SDI − TA) and their long-time behavior in order to solve (P). If the diffusion term vanishes with time, one would expect to solve (P) with our dynamics and obtain for (SDI − TA) the hierarchical minimization property described above. We refer the reader to [1] for some observations on why this happens.

Motivated by this, our paper will primarily focus on the case where $\sigma(\cdot, x)$ vanishes sufficiently fast as $t \to +\infty$ uniformly in $x$. Additionally, we will provide some guarantees for uniformly bounded $\sigma$. Therefore, throughout the paper, we assume that $\sigma$ satisfies:

$$\begin{cases} \sup_{t \geq t_0, x \in \mathbb{H}} \|\sigma(t,x)\|_{\mathrm{HS}} < +\infty, \\ \|\sigma(t,x') - \sigma(t,x)\|_{\mathrm{HS}} \leq L_0 \|x' - x\|, \end{cases} \tag{H}$$

for some $L_0 > 0$ and for all $t \geq t_0, x, x' \in \mathbb{H}$ (where HS is defined in Section 2). The Lipschitz continuity assumption is mild and required to ensure the well-posedness of (SDI) and (SDI − TA).

## 1.2 Contributions

This work goes well beyond that of [1] in three directions: we consider the non-smooth case, in infinite dimensional Hilbert spaces, and with Tikhonov regularization. The latter makes it possible to pass from weak convergence to strong convergence, and to a particular solution, that of minimal norm.

We first study the properties of the process $X(t)$ and $F(X(t))$ for the stochastic differential inclusion (SDI) on separable real Hilbert spaces from an optimization perspective, under the assumptions ($\mathrm{H}_0$), (H) and ($H_\lambda$) (introduced in Section 3). When the diffusion term is uniformly bounded, we show convergence of $\mathbb{E}[F(X(t)) - \min F]$ to a noise-dominated region both for the convex and strongly convex case. When the diffusion term is square-integrable, we show in Theorem 3.6 that $X(t)$ weakly converges almost surely to a solution of (P), which is a new result to the best of our knowledge. Moreover, in Theorem 3.8, we provide new ergodic and pointwise convergence rates of the objective in expectation, again, for both the convex and strongly convex case.

Next, we consider (SDI − TA), obtained by adding a Tikhonov regularization term to (SDI). We show in Theorem 4.1 that under certain conditions on the regularization term, $X(t)$ strongly converges almost surely to the minimum norm solution. Then, we show in Theorem 4.8 some practical situations where one can obtain an explicit form of the Tikhonov regularizer. Moreover, in Theorem 4.11, we show new convergence rates of the objective and the trajectory in expectation for the smooth case.

Tables 1 summarizes the convergence rates obtained for $\mathbb{E}[F(X(t)) - \min F]$. We use the following notation, $F = f + g$, $\sigma_* > 0$ and $\sigma_\infty(\cdot)$ is defined as

$$\sigma_\infty(t) \stackrel{\text{def}}{=} \sup_{x \in \mathbb{H}} \|\sigma(t,x)\|_{\mathrm{HS}} \quad \text{where} \quad \|\sigma(t,x)\|^2_{\mathrm{HS}} \leq \sigma_*^2, \quad \forall t \geq t_0, \forall x \in \mathbb{H}. \tag{1.1}$$

We also denote $\mathrm{EB}^p(\mathcal{S})$ the local Error Bound Inequality defined in (4.10).

| **Property of $F$** | **DI** | **SDI** ($\sup_{t \geq t_0} \sigma_\infty(t) \leq \sigma_*$) | **SDI** ($\sigma_\infty \in \mathrm{L}^2([t_0, +\infty[)$) |
|---|---|---|---|
| Convex | $t^{-1}$ | $t^{-1} + \sigma_*^2$ | $t^{-1}$ |
| $\mu$-Strongly Convex | $e^{-2\mu t}$ | $e^{-\mu t} + \sigma_*^2$ | $\max\{e^{-\mu t}, \sigma_\infty^2(t)\}$ |

Table 1: Summary of convergence rates obtained for $\mathbb{E}[F(X(t)) - \min F]$.

In Table 2, we summarize the results obtained in the smooth case for the dynamics with Tikhonov regularization, i.e., when $g \equiv 0$.

| **Property of** $f$ | **DI-TA** ($\varepsilon(t) = t^{-r}, r \in ]0, 1[$) | **SDI-TA** $\left(\varepsilon(t) = t^{-r}, r \in ]\frac{2p}{2p+1}, 1[\right)$ |
|---|---|---|
| Convex $\cap$ $\mathrm{EB}^p(\mathcal{S})$ | $t^{-r}$ | $t^{-r}$ [1] |

Table 2: Summary of convergence rates obtained for $\mathbb{E}[f(X(t)) - \min f]$ for the dynamics with Tikhonov regularization when $\varepsilon(t) = t^{-r}$.

### 1.3 Relation to prior work

The subgradient flow dynamic (DI), which is valid on a general real Hilbert space, is a dissipative dynamical system, whose study dates back to Cauchy [3]. It plays a fundamental role in optimization: it transforms the problem of minimizing $F$ into the study of the asymptotic behavior of the trajectories of (DI). Its Euler forward discretization (with stepsize $\gamma_k > 0$) is the subgradient method

$$x_{k+1} \in x_k - \gamma_k \partial F(x_k). \tag{Sub-G}$$

Or equivalently,

$$x_{k+1} = x_k - \gamma_k g_k, \tag{1.2}$$

where $g_k \in \partial F(x_k)$ for every $k \in \mathbb{N}$.

Let us focus on the finite-dimensional case ($\mathbb{H} = \mathbb{R}^d$). In [4, 5] they give conditions on the function and the stepsize to converge to within some range of the optimal value and to the optimal value. Despite (Sub-G) being a classical algorithm to solve the non-smooth convex minimization problem, it is not recommended for general use, as discussed in [6, 7]. Moreover, with the need to handle large-scale problems (such as in various areas of data science and machine learning), it has become necessary to find ways to get around the high computational cost per iteration that these problems entail. The Robbins-Monro stochastic approximation algorithm [8] is at the heart of Stochastic Gradient Descent methods, which, roughly speaking, consists in cheaply and randomly approximating the gradient at the price of obtaining a random noise in the solutions. In [9] they propose the natural generalization to the non-smooth setting, the stochastic subgradient method (S-Sub-G) that updates the iterates according to

$$x_{k+1} \in x_k - \gamma_k (\partial F(x_k) + \xi_k), \tag{S-Sub-G}$$

where $\xi_k$ denotes the (random) noise term on the subgradient at the $k$-th iteration, and $\mathbb{E}[\xi_k] = 0$.

The SDI continuous-time approach is motivated by its relations to (S-Sub-G), where the latter can be viewed as an Euler forward time discretization, and the noise $\xi_k \sim \mathcal{N}(0, \sigma_k I_d)$ (hence not necessarily bounded). The advantage of the continuous-time perspective is that it offers a deep insight and unveils the key properties of the dynamic, without being tied to a specific discretization.

We extend the work of [1] to the case where the objective is "smooth+non-smooth", being able to show the almost sure weak convergence of the trajectory to the set of minimizers and new convergence rates for the objective in the convex and strongly convex case.

[1] Whenever $\sigma_\infty^2(t) = \mathcal{O}(t^{-2r})$.

Besides, based on the work of [10], we add a Tikhonov term that let us obtain the almost sure strong convergence of the trajectory to the minimal norm solution. Moreover, we extend the convergence rates shown in [10, Theorem 5] to the stochastic case.

### 1.4 Organization of the paper

Section 2 introduces notations and reviews some necessary material from convex and stochastic analysis. Section 3 states our main convergence results of (SDI) in the case of a convex objective function under ($\mathrm{H}_0$) and with an extra assumption on the non-smooth term. We first show the almost sure weak convergence of the process towards the set of minimizers when the diffusion term is square-integrable, then we establish convergence rates for the values. Section 4 introduces an extra vanishing term called Tikhonov regularizer that let us obtain the almost sure strong convergence of (SDI − TA) to the minimal norm solution. Then we give some practical situations where we can obtain an explicit tuning of the Tikhonov regularizer. Finally in this section, we present convergence rates for the values and for the trajectory in the smooth case. Technical lemmas and theorems that are needed throughout the paper will be collected in the appendix A.

## 2 Notation and Preliminaries

We will use the following shorthand notations: Given $n \in \mathbb{N}$, $[n] \stackrel{\text{def}}{=} \{1, \ldots, n\}$. Consider $\mathbb{H}, \mathbb{K}$ real separable Hilbert spaces endowed with the inner product $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ and $\langle \cdot, \cdot \rangle_{\mathbb{K}}$, respectively, and norm $\| \cdot \|_{\mathbb{H}} = \sqrt{\langle \cdot, \cdot \rangle_{\mathbb{H}}}$ and $\| \cdot \|_{\mathbb{K}} = \sqrt{\langle \cdot, \cdot \rangle_{\mathbb{K}}}$, respectively (we omit the subscripts $\mathbb{H}$ and $\mathbb{K}$ for the sake of clarity). $I_{\mathbb{H}}$ is the identity operator from $\mathbb{H}$ to $\mathbb{H}$. $\mathcal{L}(\mathbb{K}; \mathbb{H})$ is the space of bounded linear operators from $\mathbb{K}$ to $\mathbb{H}$, $\mathcal{L}_1(\mathbb{K})$ is the space of trace-class operators, and $\mathcal{L}_2(\mathbb{K}; \mathbb{H})$ is the space of bounded linear Hilbert-Schmidt operators from $\mathbb{K}$ to $\mathbb{H}$. For $M \in \mathcal{L}_1(\mathbb{K})$, is trace is defined by

$$\operatorname{tr}(M) \stackrel{\text{def}}{=} \sum_{i \in I} \langle M e_i, e_i \rangle < +\infty,$$

where $I \subseteq \mathbb{N}$ and $(e_i)_{i \in I}$ is an orthonormal basis of $\mathbb{K}$. Besides, for $M \in \mathcal{L}(\mathbb{K}; \mathbb{H})$, $M^\star \in \mathcal{L}(\mathbb{H}; \mathbb{K})$ is the adjoint operator of $M$, and for $M \in \mathcal{L}_2(\mathbb{K}; \mathbb{H})$,

$$\|M\|_{\mathrm{HS}} \stackrel{\text{def}}{=} \sqrt{\operatorname{tr}(M M^\star)} < +\infty$$

is its Hilbert-Schmidt norm (in the finite-dimensional case is equivalent to the Frobenius norm). We denote by $\mathrm{w\text{-}lim}$ (resp. $\mathrm{s\text{-}lim}$) the limit for the weak (resp. strong) topology of $\mathbb{H}$. The notation $A : \mathbb{H} \rightrightarrows \mathbb{H}$ means that $A$ is a set-valued operator from $\mathbb{H}$ to $\mathbb{H}$. Consider $f : \mathbb{H} \to \mathbb{R}$, the sublevel of $f$ at height $r \in \mathbb{R}$ is denoted $[f \le r] \stackrel{\text{def}}{=} \{x \in \mathbb{H} : f(x) \le r\}$. For $1 \le p \le +\infty$, $\mathrm{L}^p([a, b])$ is the space of measurable functions $g : \mathbb{R} \to \mathbb{R}$ such that $\int_a^b |g(t)|^p dt < +\infty$, with the usual adaptation when $p = +\infty$. On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathrm{L}^p(\Omega; \mathbb{H})$ denotes the (Bochner) space of $\mathbb{H}$-valued random variables whose $p$-th moment (with respect to the measure $\mathbb{P}$) is finite. Other notations will be explained when they first appear.

Let us recall some important definitions and results from convex analysis; for a comprehensive coverage, we refer the reader to [11].

We denote by $\Gamma_0(\mathbb{H})$ the class of proper lsc and convex functions on $\mathbb{H}$ taking values in $\mathbb{R} \cup \{+\infty\}$. For $\mu > 0$, $\Gamma_\mu(\mathbb{H}) \subset \Gamma_0(\mathbb{H})$ is the class of $\mu$-strongly convex functions, roughly speaking, this means that there exists a quadratic lower bound on the growth of these functions. We denote by $C^s(\mathbb{H})$ the class of $s$-times continuously differentiable functions on $\mathbb{H}$. For $L \ge 0$, $C_L^{1,1}(\mathbb{H}) \subset C^1(\mathbb{H})$ is the set of functions on $\mathbb{H}$

whose gradient is $L$-Lipschitz continuous, and $C_L^2(\mathbb{H})$ is the subset of $C_L^{1,1}(\mathbb{H})$ whose functions are twice differentiable.

The *subdifferential* of a function $f \in \Gamma_0(\mathbb{H})$ is the set-valued operator $\partial f : \mathbb{H} \rightrightarrows \mathbb{H}$ such that, for every $x$ in $\mathbb{H}$,

$$\partial f(x) = \{u \in \mathbb{H} : f(y) \geq f(x) + \langle u, y - x\rangle \quad \forall y \in \mathbb{H}\}.$$

When $f$ is continuous, $\partial f(x)$ is a non-empty convex and compact set for every $x \in \mathbb{H}$. If $f$ is differentiable, then $\partial f(x) = \{\nabla f(x)\}$. For every $x \in \mathbb{H}$ such that $\partial f(x) \neq \emptyset$, the minimum norm selection of $\partial f(x)$ is the unique element $\{\partial^0 f(x)\} \stackrel{\text{def}}{=} \operatorname{argmin}_{u\in\partial f(x)} \|u\|$.

The projection of a point $x \in \mathbb{H}$ onto a closed convex set $C \subseteq \mathbb{H}$ is denoted by $\mathrm{P}_C(x)$.

## 2.1 Deterministic results on the subgradient flow with Tikhonov regularization

Let us first recall some basic facts about the deterministic case. To solve (P), a fundamental dynamic to consider is the subgradient flow of $F$, *i.e.* the following differential inclusion:

$$\dot{x}(t) \in -\partial F(x(t)). \tag{DI}$$

It is well known since the founding papers of Brezis, Baillon, and Bruck in the 1970s that, if the solution set $\operatorname{argmin}(F)$ of (P) is non-empty and $F$ is convex, lower semicontinuous (lsc) and proper, then each solution trajectory of (DI) converges weakly, and its weak limit belongs to $\operatorname{argmin}(F)$.

In general, the limit solution depends on the initial data and is a priori difficult to specify when one has a set of solutions not reduced to only one element. To remedy this difficulty we consider the differential inclusion with vanishing Tikhonov regularization, $\varepsilon(t) \to 0$ (denoted (DI-TA)) which gives

$$\dot{x}(t) + \partial F(x(t)) + \varepsilon(t)x(t) \ni 0. \tag{DI − TA}$$

To analyze the convergence properties of this dynamic, let us recall basic facts concerning the Tikhonov approximation (1963). It consists in approximating the convex minimization problem (possibly ill-posed)

$$(\mathcal{P}) \quad \min\{F(x) : \ x \in \mathcal{H}\},$$

by the strongly convex minimization problem ($\varepsilon > 0$)

$$(\mathcal{P})_\varepsilon \quad \min\left\{F(x) + \frac{\varepsilon}{2}\|x\|^2 : \ x \in \mathcal{H}\right\}$$

whose unique solution is denoted by $x_\varepsilon$. The following result was first obtained by Browder in 1966 [12, 13].

**Theorem 2.1.** *(Hierarchical minimization). Suppose that* $\mathcal{S}_F = \operatorname{argmin}(F) \neq \emptyset$. *Then,*

*(i)* $\lim_{\varepsilon\to 0} \|x_\varepsilon - x^\star\| = 0$ *where* $x^\star = \mathrm{P}_{\mathcal{S}_F}(0)$.

*(ii)* $\|x_\varepsilon\| \leq \|x^\star\|$ *for all* $\varepsilon > 0$.

The system (DI − TA) is a special case of the general dynamic model

$$\dot{x}(t) + \nabla F(x(t)) + \varepsilon(t)\nabla\Psi(x(t)) \ni 0 \tag{2.1}$$

which involves two functions $F$ and $\Psi$ intervening with different time scale. When $\varepsilon(\cdot)$ tends to zero moderately slowly, it was shown in [14] that the trajectories of (2.1) converge asymptotically to equilibria that are solutions of the following hierarchical problem: they minimize the function $\Psi$ on the set of minimizers of $F$.

The continuous and discrete-time versions of these systems have a natural connection to the best response dynamics for potential games, domain decomposition for PDE's, optimal transport, and coupled wave equations. In the case of the Tikhonov approximation, a natural choice is to take $\Psi(x) = \|x - x_d\|^2$ where $x_d$ is a desired state. By doing so, we obtain asymptotically the closest possible solution to $x_d$. By translation, we can immediately reduce ourselves to the case $x_d = 0$, as considered in this article.

The following theorem establishes the convergence of the trajectories of (DI − TA) towards the minimum norm solution under minimal assumptions on the parameter $\varepsilon(t)$.

**Theorem 2.2.** (Cominetti-Peypouquet-Sorin, [2])

*Suppose that* $\varepsilon : [t_0, +\infty[ \to \mathbb{R}_+$ *is a measurable function that satisfies:*

*(i)* $\varepsilon(t) \to 0$ *as* $t \to +\infty$*;*

*(ii)* $\displaystyle\int_{t_0}^{+\infty} \varepsilon(t)dt = +\infty.$

*Let* $x(\cdot)$ *be a solution trajectory of the continuous dynamic* (DI − TA)*. Then,* $\text{s-lim}_{t\to+\infty} x(t) = x^\star \overset{\text{def}}{=} \mathrm{P}_{\mathcal{S}_F}(0)$.

**Proof.** Set $F_\varepsilon(x) := F(x) + \frac{\varepsilon}{2}\|x\|^2$. Then (DI − TA) can be written equivalently in a denser form as

$$\dot{x}(t) + \partial F_{\varepsilon(t)}(x(t)) \ni 0.$$

Set $h(t) := \frac{1}{2}\|x(t) - x^\star\|^2$ where $x^\star = \mathrm{P}_{\mathcal{S}_F}(0)$. Derivation of $f$ and constitutive equation (DI − TA) give

$$\dot{h}(t) + \langle -\dot{x}(t), x(t) - x^\star \rangle = 0, \tag{2.2}$$

where $-\dot{x}(t) \in \partial F_{\varepsilon(t)}(x(t))$. By strong convexity of $F_{\varepsilon(t)}$, we get

$$F_{\varepsilon(t)}(x^\star) \geq F_{\varepsilon(t)}(x(t)) + \langle y(t), x^\star - x(t) \rangle + \frac{\varepsilon(t)}{2}\|x(t) - x^\star\|^2,$$

for every $y(t) \in \partial F_{\varepsilon(t)}(x(t))$.
Using that $F_{\varepsilon(t)}(x(t)) \geq F_{\varepsilon(t)}(x_{\varepsilon(t)})$, we get

$$F(x^\star) + \frac{\varepsilon(t)}{2}\|x^\star\|^2 \geq F(x_{\varepsilon(t)}) + \frac{\varepsilon(t)}{2}\|x_{\varepsilon(t)}\|^2 + \langle y(t), x^\star - x(t) \rangle + \frac{\varepsilon(t)}{2}\|x(t) - x^\star\|^2,$$

for every $y(t) \in \partial F_{\varepsilon(t)}(x(t))$ .
From $F(x^\star) \leq F(x_{\varepsilon(t)})$ we deduce

$$\langle y(t), x(t) - x^\star \rangle \geq \varepsilon(t)h(t) + \frac{\varepsilon(t)}{2}\left(\|x_{\varepsilon(t)}\|^2 - \|x^\star\|^2\right), \tag{2.3}$$

for every $y(t) \in \partial F_{\varepsilon(t)}(x(t))$.
Combining (2.2) with (2.3) we obtain

$$\dot{h}(t) + \varepsilon(t)h(t) \leq \tfrac{1}{2}\varepsilon(t)\left(\|x^\star\|^2 - \|x_{\varepsilon(t)}\|^2\right).$$

Integrate the above inequality from $t_0$ to $t$. With $m(t) := \exp \int_{t_0}^t \varepsilon(s)ds$ we get

$$h(t) \leq \frac{h(t_0)}{m(t)} + \frac{1}{2m(t)} \int_{t_0}^t m'(s) \left(\|x^\star\|^2 - \|x_{\varepsilon(s)}\|^2\right) ds. \tag{2.4}$$

According to hypothesis (i) and the classical property of the Tikhonov approximation we have $x_{\varepsilon(t)} \to x^\star$, and hence $\|x^\star\|^2 - \|x_{\varepsilon(s)}\|^2 \to 0$. To pass to the limit on (2.4) we use hypothesis (ii) which tells us that $m(t) \to +\infty$. Let us complete the argument by using that convergence implies ergodic convergence. Precisely, given $\delta > 0$, let $t_\delta > t_0$ such that $|\|x^\star\|^2 - \|x_{\varepsilon(s)}\|^2| \leq \delta$ for $s \geq t_\delta$. Then split the integral as follows

$$\begin{aligned} h(t) &\leq \frac{h(t_0)}{m(t)} + \frac{1}{2m(t)} \int_{t_0}^{t_\delta} m'(s) \left(\|x^\star\|^2 - \|x_{\varepsilon(s)}\|^2\right) ds + \delta \frac{1}{2m(t)} \int_{t_\delta}^t m'(s)ds && (2.5) \\ &\leq \frac{h(t_0)}{m(t)} + \frac{1}{2m(t)} \int_{t_0}^{t_\delta} m'(s) \left(\|x^\star\|^2 - \|x_{\varepsilon(s)}\|^2\right) ds + \frac{\delta}{2}. && (2.6) \end{aligned}$$

Then let $t$ tend to infinity, to get $\limsup_{t\to+\infty} h(t) \leq \frac{\delta}{2}$. This being true for any $\delta > 0$ gives the result. $\square$

## 2.2 Stochastic differential equations

As said before, in many cases, the drift term is subject to noise. In such a scenario, one can model these errors using a stochastic integral with respect to the measure defined by a continuous Itô martingale. In the smooth case without Tikhonov regularization, this approach has been well documented in Maulen-Soto, Fadili, Attouch [1]. This concerns the following stochastic differential equation as the stochastic counterpart of the gradient flow, let $t_0 \geq 0$ and initial data $X_0 \in \mathrm{L}^\nu(\Omega; \mathbb{H})$ (for some $\nu \geq 2$):

$$\begin{cases} dX(t) = -\nabla f(X(t))dt + \sigma(t, X(t))dW(t), \quad t > t_0 \\ X(t_0) = X_0. \end{cases} \tag{SDE}$$

Let us make precise the ingredients of this stochastic differential equation. It is defined over a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$, where the diffusion (volatility) term $\sigma : [t_0, +\infty[\times \mathbb{H} \to \mathcal{L}_2(\mathbb{K}; \mathbb{H})$ is a measurable function, and $W$ is a $\mathbb{K}$-valued Brownian motion.

Throughout this article, the diffusion term $\sigma$ is assumed to satisfy (H). In connection with this assumption, let us define $\sigma_* > 0$ and $\sigma_\infty(\cdot)$ by

$$\|\sigma(t, x)\|_{\mathrm{HS}}^2 \leq \sigma_*^2, \quad \forall t \geq 0, \forall x \in \mathbb{H}, \quad \sigma_\infty(t) \stackrel{\text{def}}{=} \sup_{x\in\mathbb{H}} \|\sigma(t, x)\|_{\mathrm{HS}}, \tag{2.7}$$

and $\sigma_\infty(\cdot)$ is a decreasing function.

Concerning the study of (SDI) and (SDI $-$ TA), let us recall the following result of [1, Theorem 3.4] on which we will build our study. It establishes almost sure weak convergence of $X(t)$ to an $\mathcal{S}$-valued random variable as $t \to +\infty$.

**Theorem 2.3.** *Consider the dynamic* (SDE) *where $f$ and $\sigma$ satisfy the assumptions* ($\mathrm{H}_0$) *and* (H)*. Let $\nu \geq 2$, and its initial data $X_0 \in \mathrm{L}^\nu(\Omega; \mathbb{H})$. Then, there exists a unique solution $X \in S_{\mathbb{H}}^\nu[t_0]$ of* (SDE)*. Additionally, if $\sigma_\infty \in \mathbb{L}^2([t_0, +\infty[)$, then:*

*(i)* $\sup_{t\geq 0} \mathbb{E}[\|X(t)\|^2] < +\infty$.

*(ii)* $\forall x^\star \in \mathcal{S}$, $\lim_{t\to+\infty} \|X(t) - x^\star\|$ *exists a.s. and* $\sup_{t\geq 0} \|X(t)\| < +\infty$ *a.s.*

*(iii)* $\lim_{t\to\infty} \|\nabla f(X(t))\| = 0$ *a.s. As a result,* $\lim_{t\to\infty} f(X(t)) = \min f$ *a.s.*

*(iv) There exists an $\mathcal{S}$-valued random variable $X^\star$ such that* $\text{w-}\lim_{t\to+\infty} X(t) = X^\star$ *a.s.*

# 3 Stochastic differential inclusions

In this section, we will work with stochastic differential inclusions. For the history of this concept, we refer the reader to [15, Preface]. We will start by showing a general version of the (SDI) dynamic, formally describing what it means to be a solution of that dynamic, and then we will move on to show the conditions under which you can have the existence and uniqueness of a solution. Existence is due to [16] and uniqueness is proven here. Then we will focus on (SDI) and study the conditions on the diffusion term in order to ensure the almost sure weak convergence of the trajectory towards the set of minimizers. Finally, we will show some convergence rates of the objective under the hypothesis of convexity or strong convexity.

## 3.1 Existence and uniqueness of solution

Let $A : \mathbb{H} \rightrightarrows \mathbb{H}$, $b : [t_0, +\infty[\times\mathbb{H} \to \mathbb{H}$ and $\sigma : [t_0, +\infty[\times\mathbb{H} \to \mathcal{L}_2(\mathbb{K}; \mathbb{H})$. Let $t_0 \geq 0$ and consider the general stochastic differential inclusion:

$$\begin{cases} dX(t) \in b(t, X(t))dt - A(X(t))dt + \sigma(t, X(t))dW(t), \quad t > 0 \\ X(t_0) = X_0, \end{cases} \tag{SDI$_0$}$$

defined over a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq t_0}, \mathbb{P})$, where the diffusion (volatility) term $\sigma : [t_0, +\infty[\times\mathbb{H} \to \mathcal{L}_2(\mathbb{K}; \mathbb{H})$ is a measurable function; $W$ is a $\mathcal{F}_t$-adapted $\mathbb{K}$-valued Brownian motion; and the initial data $X_0$ is an $\mathcal{F}_0$-measurable $\mathbb{H}$-valued random variable.

**Definition 3.1.** A solution of (SDI$_0$) is a couple $(X, \eta)$ of $\mathcal{F}_t$-adapted processes such that almost surely:

(i) $X$ is continuous with sample paths in the domain of $A$;

(ii) $\eta$ is absolutely continuous, such that $\eta(t_0) = 0$, and $\forall T > t_0$, $\eta' \in \mathrm{L}^2([t_0, T]; \mathbb{H})$, $\eta'(t) \in A(X(t))$ for almost all $t \geq t_0$;

(iii) For $t > t_0$,

$$\begin{cases} X(t) &= X_0 + \int_{t_0}^t b(s, X(s))ds - \eta(t) + \int_{t_0}^t \sigma(s, X(s))dW(s), \\ X(t_0) &= X_0. \end{cases} \tag{3.1}$$

For brevity, we sometimes omit the process $\eta$ and say that $X$ is a solution of (SDI$_0$), meaning that, there exists a process $\eta$ such that $(X, \eta)$ satisfies the previous definition.

The definition of uniqueness for the process $X$ will be presented in Section A.2.1.

Throughout the paper it will be assumed:

$$\begin{cases} A \text{ is a maximal monotone operator with closed domain;} \\ \mathcal{S} \stackrel{\text{def}}{=} A^{-1}(0) \neq \emptyset. \end{cases} \tag{H$_0$(A)}$$

$$\begin{cases} \exists L > 0, \|b(t,x) - b(t,y)\| \vee \|\sigma(t,x) - \sigma(t,y)\|_{\mathrm{HS}} \leq L\|x - y\|, \forall t \geq t_0, \forall x, y \in \mathbb{H}; \\ \sup_{t\geq t_0}(\|b(t,0)\| \vee \|\sigma(t,0)\|_{\mathrm{HS}}) < +\infty. \end{cases} \tag{H$_0$(b,$\sigma$)}$$

The Lipschitz continuity assumption is mild and required to ensure the well-posedness of (SDI$_0$).

We are interested in ensuring the existence and uniqueness of a solution for (SDI$_0$). Although there are several works that deal with the subject of stochastic differential inclusions (see [15, 17, 18, 19, 16, 20]), those of [16, 20] are the closest to our setting and define a solution in the sense of Definition 3.1, thus generalizing the work of Brezis [21] in the deterministic case to the stochastic setting. In this paper, we consider the sequence of solutions $\{X_\lambda\}_{\lambda>0}$ of the stochastic differential equations

$$\begin{cases} dX_\lambda(t) = b(t, X(t))dt - A_\lambda(X_\lambda(t))dt + \sigma(t, X_\lambda(t))dW(t), & t > t_0 \\ X_\lambda(t_0) = X_0, \end{cases} \tag{SDE$_\lambda$}$$

where $A_\lambda = (I - (I + \lambda A)^{-1})/\lambda$ is the Yosida approximation of $A$ with parameter $\lambda > 0$. Under the integrability condition

$$\limsup_{\lambda\downarrow 0} \int_{t_0}^{T} \mathbb{E}(\|A_\lambda(X_\lambda(t))\|^2)dt < +\infty, \tag{$H_\lambda$}$$

it was shown in [16] that there exists a couple $(X, \eta)$ of stochastic processes such that for every $T > t_0$,

$$\lim_{\lambda\downarrow 0} \mathbb{E}\left(\sup_{t\in[t_0,T]} \|X_\lambda(t) - X(t)\|^2\right) = 0, \quad \lim_{\lambda\downarrow 0} \mathbb{E}\left(\sup_{t\in[t_0,T]} \|\eta_\lambda - \eta\|^2\right) = 0,$$

where $\eta_\lambda(t) = \int_{t_0}^t A_\lambda(X_\lambda(s))ds$, and that $(X, \eta)$ is a solution of (SDI$_0$) in the sense of Definition 3.1. Moreover, one can even have a.s. convergence of the process $X_\lambda$ when the diffusion term is state-independent; see [16, Proposition 6.3].

**Remark 3.2.** Condition ($H_\lambda$) is satisfied under different conditions, some examples are mentioned in [16]. One case where this condition holds is when $A$ is full domain and there exists $C_0 > 0$ such that $\|A^0(x)\| \le C_0(1 + \|x\|)$ for $x \in \mathbb{H}$, where $A^0(x) = \operatorname{argmin}_{y\in A(x)} \|y\|$.

**Theorem 3.3.** *Consider* (SDI$_0$)*, where $A$ and $(b, \sigma)$ satisfy the assumption* (H$_0$(A)) *and* (H$_0$(b, $\sigma$))*, respectively. Additionally, suppose that $A$ satisfy* ($H_\lambda$) *and let $\nu \ge 2$ such that $X_0 \in \mathrm{L}^\nu(\Omega; \mathbb{H})$ and is $\mathcal{F}_0$-measurable. Then, there exists a unique solution $(X, \eta) \in S^\nu_{\mathbb{H}}[t_0] \times C^1([t_0, +\infty[; \mathbb{H})$ of* (SDI$_0$).

**Proof.** The existence of a solution $(X, \eta)$ in the sense of Definition 3.1 comes from [16, Theorem 3.5]. We now turn to uniqueness. Let $(X_1, \eta_1)$ and $(X_2, \eta_2)$ be two solutions of (SDI$_0$). By Itô's formula, we have

$$\begin{aligned} \|X_1(t) - X_2(t)\|^2 &= 2\int_{t_0}^t \langle b(s, X_1(s)) - b(s, X_2(s)), X_1(s) - X_2(s)\rangle ds \\ &\quad - 2\int_{t_0}^t \langle \eta_1'(s) - \eta_2'(s), X_1(s) - X_2(s)\rangle ds + \int_{t_0}^t \|\sigma(s, X_1(s)) - \sigma(s, X_2(s))\|^2_{\mathrm{HS}} ds \\ &\quad + \int_{t_0}^t \langle X_1(s) - X_2(s), [\sigma(s, X_1(s)) - \sigma(s, X_2(s))]dW(s)\rangle. \end{aligned}$$

Since for almost all $t \ge 0$, $\eta_i'(t) \in A(X_i(t))$, $i = \{1, 2\}$, by monotonicity of $A$, we have that for almost all $t \ge t_0$,

$$\langle \eta_1'(t) - \eta_2'(t), X_1(t) - X_2(t)\rangle \ge 0,$$

and thus the second term on the right-hand side is non-positive. Now, let $n \in \mathbb{N}$ arbitrary and consider the stopping time $\tau_n = \inf\{t \ge t_0 : \|X_1(t) - X_2(t)\| \ge n\}$ and evaluate the previous equation at $t \wedge \tau_n$, denoting $X_i^n(t) = X_i(t \wedge \tau_n)$ $(i = \{1, 2\})$, we have

$$\begin{aligned}
\|X_1^n(t) - X_2^n(t)\|^2 \leq 2 &\int_{t_0}^{t\wedge\tau_n} \langle b(s, X_1(s)) - b(s, X_2(s)), X_1(s) - X_2(s)\rangle ds \\
&+ \int_{t_0}^{t\wedge\tau_n} \|\sigma(s, X_1(s)) - \sigma(s, X_2(s))\|_{\mathrm{HS}}^2 ds \\
&+ \int_{t_0}^{t\wedge\tau_n} \langle X_1(s) - X_2(s), [\sigma(s, X_1(s)) - \sigma(s, X_2(s))]dW(s)\rangle \\
&\leq L(L+2) \int_{t_0}^{t\wedge\tau_n} \|X_1(s) - X_2(s)\|^2 ds \\
&+ \int_{t_0}^{t\wedge\tau_n} \langle X_1(s) - X_2(s), [\sigma(s, X_1(s)) - \sigma(s, X_2(s))]dW(s)\rangle \\
&\leq L(L+2) \int_{t_0}^{t} \|X_1^n(s) - X_2^n(s)\|^2 ds \\
&+ \int_{t_0}^{t\wedge\tau_n} \langle X_1(s) - X_2(s), [\sigma(s, X_1(s)) - \sigma(s, X_2(s))]dW(s)\rangle.
\end{aligned}$$

Note that we have used Cauchy-Schwarz inequality and the Lipschitz assumption on $(b, \sigma)$ in the second inequality. Taking expectation of both sides and using the properties of Itô's integral we obtain

$$\mathbb{E}(\|X_1^n(t) - X_2^n(t)\|^2) \leq L(L+2) \int_{t_0}^{t} \mathbb{E}(\|X_1^n(s) - X_2^n(s)\|^2) ds.$$

By Grönwall's inequality, we obtain that

$$\mathbb{E}(\|X_1^n(t) - X_2^n(t)\|^2) = 0, \forall t \geq t_0, \forall n \in \mathbb{N}.$$

On the other hand, we have that $\lim_{n\to+\infty} t \wedge \tau_n = t$. Therefore, taking $\liminf_{n\to+\infty}$ in the previous expression, using Fatou's Lemma and the fact that $X_1, X_2$ are a.s. continuous processes, we conclude that $\mathbb{E}(\|X_1(t) - X_2(t)\|^2) = 0$, consequently

$$\mathbb{P}(X_1(t) = X_2(t), \forall t \in [t_0, T]) = 1, \quad \text{for every } T > t_0.$$

Let $T > t_0$ arbitrary, let us prove that $\mathbb{E}\left(\sup_{t\in[t_0,T]} \|X(t)\|^\nu\right) < +\infty$. Using Itô's formula with the solution process $X$ and the anchor function $\phi(x) = \|x - x^\star\|^2$ for $x^\star \in A^{-1}(0)$, we obtain for every $t \in [t_0, T]$:

$$\begin{aligned}
\|X(t) - x^\star\|^2 = \|X_0 - x^\star\|^2 + 2\int_{t_0}^{t} \langle b(s, X(s)), X(s) - x^\star\rangle ds - 2\int_{t_0}^{t} \langle \eta'(s), X(s) - x^\star\rangle ds \\
+ \int_{t_0}^{t} \|\sigma(s, X(s))\|_{\mathrm{HS}}^2 ds + 2\int_{t_0}^{t} \langle X(s) - x^\star, \sigma(s, X(s))dW(s)\rangle.
\end{aligned}$$

Since $\eta'(t) \in A(X(t))$ for almost all $t \geq 0$, and $0 \in A(x^\star)$, by monotonicity of $A$ we have that for every $t \in [t_0, T]$,

$$\langle \eta'(t), X(t) - x^\star\rangle \geq 0, \quad \text{for almost all } t \geq 0.$$

Thus the second integral is nonnegative, which implies

$$\begin{aligned}\|X(t)-x^\star\|^2 \le \|X_0-x^\star\|^2 + 2\int_{t_0}^t \langle b(s,X(s)), X(s)-x^\star\rangle ds + \int_{t_0}^t \|\sigma(s,X(s))\|_{\mathrm{HS}}^2 ds \\ + 2\int_{t_0}^t \langle X(s)-x^\star, \sigma(s,X(s))dW(s)\rangle.\end{aligned} \tag{3.2}$$

Moreover, we have

$$2\langle b(t,x), x-x^\star\rangle + \|\sigma(t,x)\|_{\mathrm{HS}}^2 \le 2\|b(t,x)\|\|x-x^\star\| + \|\sigma(t,x)\|_{\mathrm{HS}}^2 \le C(1+\|x-x^\star\|^2), \quad \forall t\ge t_0, \forall x\in\mathbb{H}.$$

We now proceed as in the proof of [22, Lemma 3.2] to conclude that $X \in S^\nu_{\mathbb{H}}[t_0]$. In fact, we take power $\frac{\nu}{2}$ at both sides of (3.2), then using that $(a+b+c)^{\frac{\nu}{2}} \le 3^{\frac{\nu-2}{2}}(a^{\frac{\nu}{2}}+b^{\frac{\nu}{2}}+c^{\frac{\nu}{2}})$ we have

$$\begin{aligned}\|X(t)-x^\star\|^\nu \le 3^{\frac{\nu-2}{2}}\left(\|X_0-x^\star\|^\nu + C^{\frac{\nu}{2}}\left(\int_{t_0}^t 1+\|X(s)-x^\star\|^2 ds\right)^{\frac{\nu}{2}}\right) \\ + 3^{\frac{\nu-2}{2}}2^{\frac{\nu}{2}}\left(\int_{t_0}^t \langle X(s)-x^\star, \sigma(s,X(s))dW(s)\rangle\right)^{\frac{\nu}{2}}.\end{aligned}$$

Now taking supremum $t\in[t_0,T]$ and then expectation at both sides, we have that there exists $K = K(\nu,T)$ such that:

$$\begin{aligned}\mathbb{E}\left(\sup_{t\in[t_0,T]}\|X(t)-x^\star\|^\nu\right) \le K\left(1+\mathbb{E}\left(\|X_0-x^\star\|^\nu\right) + \int_{t_0}^T \mathbb{E}(\|X(s)-x^\star\|^\nu)ds\right) \\ + K\mathbb{E}\left(\sup_{t\in[t_0,T]}\left|\int_{t_0}^t \langle X(s)-x^\star, \sigma(s,X(s))dW(s)\rangle\right|^{\frac{\nu}{2}}\right).\end{aligned}$$

By Proposition A.5, we get that, for a redefined $K = K(\nu,T)$,

$$\begin{aligned}\mathbb{E}\left(\sup_{t\in[t_0,T]}\|X(t)-x^\star\|^\nu\right) \le K\left(1+\mathbb{E}\left(\|X_0-x^\star\|^\nu\right) + \int_{t_0}^T \mathbb{E}(\|X(s)-x^\star\|^\nu)ds\right) \\ + K\mathbb{E}\left(\left|\int_{t_0}^T \|X(s)-x^\star\|^2\|\sigma(s,X(s))\|_{\mathrm{HS}}^2 ds\right|^{\frac{\nu}{4}}\right).\end{aligned} \tag{3.3}$$

Note that by Cauchy-Schwarz and Young's inequality,

$$\begin{aligned}&\mathbb{E}\left(\left|\int_{t_0}^T \|X(s)-x^\star\|^2\|\sigma(s,X(s))\|_{\mathrm{HS}}^2 ds\right|^{\frac{\nu}{4}}\right) \\ &\le \mathbb{E}\left(\sup_{t\in[t_0,T]}\|X(t)-x^\star\|^{\frac{\nu}{2}}\left(\int_{t_0}^T \|\sigma(s,X(s))\|_{\mathrm{HS}}^2\right)^{\frac{\nu}{4}}\right) \\ &\le \frac{1}{2K}\mathbb{E}\left(\sup_{t\in[t_0,T]}\|X(t)-x^\star\|^\nu\right) + \frac{K}{2}\mathbb{E}\left[\left(\int_{t_0}^T \|\sigma(s,X(s))\|_{\mathrm{HS}}^2\right)^{\frac{\nu}{2}}\right] \\ &\le \frac{1}{2K}\mathbb{E}\left(\sup_{t\in[t_0,T]}\|X(t)-x^\star\|^\nu\right) + \frac{KC^{\frac{\nu}{2}}}{2}\mathbb{E}\left[\left(\int_{t_0}^T 1+\|X(s)-x^\star\|^2 ds\right)^{\frac{\nu}{2}}\right] \\ &\le \frac{1}{2K}\mathbb{E}\left(\sup_{t\in[t_0,T]}\|X(t)-x^\star\|^\nu\right) + \frac{KC^{\frac{\nu}{2}}}{2}T^{\frac{\nu-2}{2}}\mathbb{E}\left[\left(\int_{t_0}^T (1+\|X(s)-x^\star\|^2)^{\frac{\nu}{2}}ds\right)\right].\end{aligned}$$

Substituting this into (3.3), we have, for a possibly different $K = K(\nu, T)$,

$$\mathbb{E}\left(\sup_{t\in[t_0,T]} \|X(t) - x^\star\|^\nu\right) \leq K\left(1 + \mathbb{E}\left(\|X_0 - x^\star\|^\nu\right) + \int_{t_0}^{T} \mathbb{E}\left(\sup_{t\in[0,s]} \|X(t) - x^\star\|^\nu\right) ds\right).$$

By Grönwall's inequality, we obtain

$$\mathbb{E}\left(\sup_{t\in[t_0,T]} \|X(t) - x^\star\|^\nu\right) \leq K\left(1 + \mathbb{E}\left(\|X_0 - x^\star\|^\nu\right)\right) e^{KT} < +\infty.$$

Since $T > t_0$ is arbitrary, we conclude that $X \in S^\nu_{\mathbb{H}}[t_0]$. $\square$

**Corollary 3.4.** *Consider* ($\mathrm{SDE}_\lambda$)*, where $A$ and $(b, \sigma)$ satisfy the assumption* ($\mathrm{H_0(A)}$) *and* ($\mathrm{H_0(b,\sigma)}$)*, respectively. Additionally, let us consider that $A$ satisfy* ($H_\lambda$) *and let $\nu \geq 2$ such that $X_0 \in \mathrm{L}^\nu(\Omega; \mathbb{H})$ and is $\mathcal{F}_0$-measurable. Then,*

$$\sup_{\lambda>0} \mathbb{E}\left(\sup_{t\in[t_0,T]} \|X_\lambda(t)\|^\nu\right) < +\infty.$$

**Proof.** Since $A^{-1}(0) = A_\lambda^{-1}(0)$ and $A_\lambda$ is monotone, we replace $\eta'$ by $A_\lambda(X_\lambda)$ in the proof of Theorem 3.3, then we realize that the constant that bounds $\mathbb{E}\left(\sup_{t\in[t_0,T]} \|X_\lambda(t)\|^\nu\right)$ is independent from $\lambda$ to conclude. $\square$

Let us present our extension of Itô's formula for a multi-valued drift, which plays a central role in the study of SDI's.

**Proposition 3.5.** *Consider* ($\mathrm{SDI}_0$) *under the assumptions of Theorem (3.3). Let $(X, \eta) \in S^\nu_{\mathbb{H}}[t_0] \times C^1([t_0, +\infty[; \mathbb{H})$ be the unique solution of* ($\mathrm{SDI}_0$)*, and let $\phi : [t_0, +\infty[ \times \mathbb{H} \to \mathbb{R}$ be such that $\phi(\cdot, x) \in C^1([t_0, +\infty[)$ for every $x \in \mathbb{H}$ and $\phi(t, \cdot) \in C^2(\mathbb{H})$ for every $t \geq t_0$. Then the process*

$$Y(t) = \phi(t, X(t)),$$

*is an Itô Process such that for all $t \geq 0$*

$$\begin{aligned} Y(t) = Y(t_0) &+ \int_{t_0}^{t} \frac{\partial\phi}{\partial t}(s, X(s))ds + \int_{t_0}^{t} \left\langle \nabla\phi(s, X(s)), b(s, X(s)) - \eta'(s)\right\rangle ds \\ &+ \int_{t_0}^{t} \langle \sigma^\star(s, X(s))\nabla\phi(s, X(s)), dW(s)\rangle + \frac{1}{2}\int_{t_0}^{t} \mathrm{tr}\left(\sigma(s, X(s))\sigma^\star(s, X(s))\nabla^2\phi(s, X(s))\right) ds, \end{aligned} \tag{3.4}$$

*where $\eta'(t) \in A(X(t))$ a.s. for almost all $t \geq t_0$. Moreover, if $\mathbb{E}[Y(t_0)] < +\infty$, and if for all $T > t_0$*

$$\mathbb{E}\left(\int_{t_0}^{T} \|\sigma^\star(s, X(s))\nabla\phi(s, X(s))\|^2 ds\right) < +\infty,$$

*then $\int_{t_0}^{t} \langle \sigma^\star(s, X(s))\nabla\phi(s, X(s)), dW(s)\rangle$ is a square-integrable continuous martingale and*

$$\begin{aligned} \mathbb{E}[Y(t)] = \mathbb{E}[Y(t_0)] &+ \mathbb{E}\left(\int_{t_0}^{t} \frac{\partial\phi}{\partial t}(s, X(s))ds\right) + \mathbb{E}\left(\int_{t_0}^{t} \left\langle \nabla\phi(s, X(s)), b(s, X(s)) - \eta'(s)\right\rangle ds\right) \\ &+ \frac{1}{2}\mathbb{E}\left(\int_{t_0}^{t} \mathrm{tr}\left(G(s, X(s))G^\star(s, X(s))\nabla^2\phi(s, X(s))\right) ds\right). \end{aligned} \tag{3.5}$$

**Proof.** The unique solution $(X,\eta) \in S^{\nu}_{\mathbb{H}}[t_0] \times C^1([t_0,+\infty[;\mathbb{H})$ of (SDI$_0$) satisfies (by definition) the following equation:

$$\begin{cases} X(t) & = X_0 + \int_{t_0}^t [b(s,X(s)) - \eta'(s)]ds + \int_{t_0}^t \sigma(s,X(s))dW(s), \quad t > t_0, \\ X(t_0) & = X_0. \end{cases} \tag{3.6}$$

and $\eta'(s) \in A(X(s))$ for almost all $t \geq 0$ a.s.. Then, (3.6) is an Itô process with drift $s \mapsto b(s,X(s)) - \eta'(s)$ and diffusion $s \mapsto \sigma(s,X(s))$. Consequently, we can apply the classical Itô's formula (see [23, Section 2.3]) to obtain the desired. ☐

### 3.2 Almost sure weak convergence of the trajectory

We consider $f+g$ (called the potential) and study the dynamic (SDI) under the hypotheses (H$_0$) (*i.e.* $f \in C_L^{1,1}(\mathbb{H}) \cap \Gamma_0(\mathbb{H}), g \in \Gamma_0(\mathbb{H})$) and (H). Recall the definitions of $\sigma_*$ and $\sigma_\infty(t)$ from (1.1). Throughout the rest of the paper, we use the notation

$$\begin{aligned} F(x) &\stackrel{\text{def}}{=} f(x) + g(x) \\ \Sigma(t,x) &\stackrel{\text{def}}{=} \sigma(t,x)\sigma(t,x)^\star \\ \mathcal{S}_F &\stackrel{\text{def}}{=} \operatorname{argmin}(F). \end{aligned}$$

Our first main result establish almost sure weak convergence of $X(t)$ to a point that belongs in $\mathcal{S}_F$. It is based on Itô's formula, and on Barbalat's and Opial's Lemma. It follows the same ideas as in [1, Theorem 3.1].

**Theorem 3.6.** *Consider $F = f + g$ and $\sigma$ satisfying (H$_0$) and (H) respectively. Suppose further that $\partial g$ verifies ($H_\lambda$). Let $\nu \geq 2$, $t_0 \geq 0$ , and consider the dynamic (SDI) with initial data $X_0 \in \mathrm{L}^\nu(\Omega;\mathbb{H})$, i.e.:*

$$\begin{aligned} &dX(t) \in -\partial F(X(t))dt + \sigma(t,X(t))dW(t) \\ &X(t_0) = X_0, \end{aligned} \tag{3.7}$$

*where $W$ is a $\mathbb{K}$-valued Brownian motion. Then, there exists a unique solution (in the sense of Theorem 3.3) $(X,\eta) \in S^{\nu}_{\mathbb{H}}[t_0] \times C^1([t_0,+\infty[;\mathbb{H})$.*

*Moreover, if $\sigma_\infty \in \mathrm{L}^2([t_0,+\infty[)$, then the following holds:*

*(i)* $\mathbb{E}[\sup_{t\geq t_0} \|X(t)\|^\nu] < +\infty$.

*(ii)* $\forall x^\star \in \mathcal{S}_F$, $\lim_{t\to+\infty} \|X(t) - x^\star\|$ *exists a.s. and* $\sup_{t\geq t_0}\|X(t)\| < +\infty$ *a.s..*

*(iii) If $g$ is continuous , then $\forall x^\star \in \mathcal{S}_F$, $\nabla f(x^\star)$ is constant, $\lim_{t\to\infty} \|\nabla f(X(t)) - \nabla f(x^\star)\| = 0$ a.s.. and*

$$\int_{t_0}^{+\infty} F(X(t)) - \min F \, dt < +\infty.$$

*(iv) If (iii) holds, then there exists an $\mathcal{S}_F$-valued random variable $X^\star$ such that* $\text{w-}\lim_{t\to+\infty} X(t) = X^\star$.

**Remark 3.7.** By classical properties of the Yosida approximation

$$(\partial g(x))_\lambda = \nabla g_\lambda(x) = \frac{1}{\lambda}(x - \operatorname{prox}_{\lambda g}(x)),$$

where $g_\lambda$ is the Moreau envelope of $g$ with parameter $\lambda > 0$. If there exists $C > 0$, such that

$$\|x - \mathrm{prox}_{\lambda g}(x)\| \leq \lambda C,$$

the assumption ($H_\lambda$) is satisfied by $\partial g$. As mentioned in Remark 3.2, if $g$ is continuous and $\|\partial^0 g(x)\| \leq C_0(1 + \|x\|)$ for some $C_0 > 0$, then $\partial g$ also satisfies ($H_\lambda$).

**Proof.** (i) Directly from Theorem 3.3.

(ii) Since $F$ is convex, we first notice that $\mathcal{S}_F = (\partial F)^{-1}(0)$.

Now let us consider $(X, \eta) \in S^\nu_{\mathbb{H}}[t_0] \times C^1([t_0, +\infty[; \mathbb{H})$ be the unique solution of ($\mathrm{SDI}_0$) given by Theorem 3.3, and $\phi(x) = \frac{\|x - x^\star\|^2}{2}$, where $x^\star \in \mathcal{S}_F$. Then by Itô's formula

$$\phi(X(t)) = \underbrace{\frac{\|X_0 - x^\star\|^2}{2}}_{\xi = \phi(X_0)} + \underbrace{\frac{1}{2}\int_{t_0}^t \mathrm{tr}\left(\Sigma(s, X(s))\right) ds}_{A_t} - \underbrace{\int_{t_0}^t \left\langle \eta'(s) + \nabla f(X(s)), X(s) - x^\star \right\rangle ds}_{U_t}$$
$$+ \underbrace{\int_{t_0}^t \left\langle \sigma^\star(s, X(s))\left(X(s) - x^\star\right), dW(s)\right\rangle}_{M_t}. \tag{3.8}$$

Let us observe that, since $\nu \geq 2$, we have that $\mathbb{E}(\sup_{t \geq t_0} \|X(t)\|^2) < +\infty$. Moreover, since $\sigma_\infty \in \mathrm{L}^2([t_0, +\infty[)$ we have

$$\mathbb{E}\left(\int_{t_0}^{+\infty} \|\sigma^\star(s, X(s))\left(X(s) - x^\star\right)\|^2 ds\right) \leq \mathbb{E}\left(\sup_{t \geq t_0} \|X(t) - x^\star\|^2\right) \int_{t_0}^{+\infty} \sigma^2_\infty(s) ds < +\infty.$$

Therefore $M_t$ is a square-integrable continuous martingale. It is also a continuous local martingale (see [24, Theorem 1.3.3]), which implies that $\mathbb{E}(M_t) = 0$.

Moreover, since $F$ is a convex function, then $\partial F$ is a monotone operator. On the other hand $\eta'(t) \in \partial g(X(t))$ a.s. for almost all $t \geq t_0$, so

$$\left\langle \eta'(t) + \nabla f(X(t)), X(t) - x^\star \right\rangle \geq 0, a.s. \text{for almost all } t \geq t_0.$$

We have that $A_t$ and $U_t$ defined as in (3.8) are two continuously adapted increasing processes with $A_0 = U_0 = 0$ a.s.. Since $\phi(X(t))$ is nonnegative and $\sup_{x \in \mathbb{H}} \|\sigma(\cdot, x)\|_{\mathrm{HS}} \in \mathrm{L}^2([t_0, +\infty[)$, we deduce that $\lim_{t \to +\infty} A_t < +\infty$. Then, we can use Theorem A.7 to conclude that

$$\int_{t_0}^{+\infty} \langle \eta'(t) + \nabla f(X(t)), X(t) - x^\star \rangle dt < +\infty \quad a.s. \tag{3.9}$$

and

$$\forall x^\star \in \mathcal{S}_F, \exists \Omega_{x^\star} \in \mathcal{F}, \text{such that } \mathbb{P}(\Omega_{x^\star}) = 1 \text{ and } \lim_{t \to +\infty} \|X(\omega, t) - x^\star\| \text{ exists } \forall \omega \in \Omega_{x^\star}. \tag{3.10}$$

Since $\mathbb{H}$ is separable, there exists a countable set $\mathcal{Z} \subseteq \mathcal{S}_F$, such that $\mathrm{cl}(\mathcal{Z}) = \mathcal{S}_F$ (where cl stands for the closure of the set). Let $\widetilde{\Omega} = \bigcap_{z \in \mathcal{Z}} \Omega_z$. Since $\mathcal{Z}$ is countable, a union bound shows

$$\mathbb{P}(\widetilde{\Omega}) = 1 - \mathbb{P}\left(\bigcup_{z \in \mathcal{Z}} \Omega_z^c\right) \geq 1 - \sum_{z \in \mathcal{Z}} \mathbb{P}(\Omega_z^c) = 1.$$

For arbitrary $x^\star \in \mathcal{S}_F$, there exists a sequence $(z_k)_{k\in\mathbb{N}} \subseteq \mathcal{Z}$ such that $\lim_{k\to\infty} z_k = x^\star$. In view of (3.10), for every $k \in \mathbb{N}$ there exists $\tau_k : \Omega_{z_k} \to \mathbb{R}_+$ such that

$$\lim_{t\to+\infty} \|X(\omega,t) - z_k\| = \tau_k(\omega), \quad \forall \omega \in \Omega_{z_k}. \tag{3.11}$$

Now, let $\omega \in \widetilde{\Omega}$. Since $\widetilde{\Omega} \subset \Omega_{z_k}$ for any $k \in \mathbb{N}$, and using the triangle inequality and (3.11), we obtain that

$$\tau_k(\omega) - \|z_k - x^\star\| \leq \liminf_{t\to+\infty} \|X(\omega,t) - x^\star\| \leq \limsup_{t\to+\infty} \|X(\omega,t) - x^\star\| \leq \tau_k(\omega) + \|z_k - x^\star\|.$$

Now, passing to $k \to +\infty$, we deduce

$$\limsup_{k\to+\infty} \tau_k(\omega) \leq \liminf_{t\to+\infty} \|X(\omega,t) - x^\star\| \leq \limsup_{t\to+\infty} \|X(\omega,t) - x^\star\| \leq \liminf_{k\to+\infty} \tau_k(\omega),$$

whence we deduce that $\lim_{k\to+\infty} \tau_k(\omega)$ exists on the set $\widetilde{\Omega}$ of probability 1, and in turn

$$\lim_{t\to+\infty} \|X(\omega,t) - x^\star\| = \lim_{k\to+\infty} \tau_k(\omega).$$

Let us recall that there exists $\Omega_{\text{cont}} \in \mathcal{F}$ such that $\mathbb{P}(\Omega_{\text{cont}}) = 1$ and $X(\omega,\cdot)$ is continuous for every $\omega \in \Omega_{\text{cont}}$. Now let $x^\star \in \mathcal{S}_F$ arbitrary, since the limit exists, for every $\omega \in \widetilde{\Omega} \cap \Omega_{\text{cont}}$ there exists $T(\omega)$ such that $\|X(\omega,t) - x^\star\| \leq 1 + \lim_{k\to+\infty} \tau_k(\omega)$ for every $t \geq T(\omega)$. Besides, since $X(\omega,\cdot)$ is continuous, by Bolzano's theorem

$$\sup_{t\in[0,T(\omega)]} \|X(\omega,t)\| = \max_{t\in[0,T(\omega)]} \|X(\omega,t)\| \stackrel{\text{def}}{=} h(\omega) < +\infty.$$

Therefore, $\sup_{t\geq t_0} \|X(\omega,t)\| \leq \max\{h(\omega), 1 + \lim_{k\to+\infty} \tau_k(\omega) + \|x^\star\|\} < +\infty$.

(iii) Let $N_t = \displaystyle\int_{t_0}^t \sigma(s,X(s))dW(s)$. This is a continuous martingale (w.r.t. the filtration $\mathcal{F}_t$), which verifies

$$\mathbb{E}(\|N_t\|^2) = \mathbb{E}\left(\int_{t_0}^t \|\sigma(s,X(s))\|_{\text{HS}}^2\, ds\right) \leq \mathbb{E}\left(\int_{t_0}^{+\infty} \sigma_\infty^2(s)ds\right) < +\infty, \forall t \geq t_0.$$

According to Theorem A.6, we deduce that there exists a $\mathbb{H}-$valued random variable $N_\infty$ w.r.t. $\mathcal{F}_\infty$, and which verifies: $\mathbb{E}(\|N_\infty\|^2) < +\infty$, and there exists $\Omega_N \in \mathcal{F}$ such that $\mathbb{P}(\Omega_N) = 1$ and

$$\lim_{t\to+\infty} N_t(\omega) = N_\infty(\omega) \text{ for every } \omega \in \Omega_N.$$

On the other hand, since $x^\star \in (\partial F)^{-1}(0) = (\nabla f + \partial g)^{-1}(0)$, then $-\nabla f(x^\star) \in \partial g(x^\star)$. Let $T > t_0$ such that $\eta'(t) \in \partial g(X(t))$ a.s., consequently,

$$\begin{aligned}\langle \eta'(t) + \nabla f(X(t)), X(t) - x^\star\rangle &= \underbrace{\langle \eta'(t) - (-\nabla f(x^\star)), X(t) - x^\star\rangle}_{\geq 0} \\ &\quad + \langle \nabla f(X(t)) - \nabla f(x^\star), X(t) - x^\star\rangle \\ &\geq \langle \nabla f(X(t)) - \nabla f(x^\star), X(t) - x^\star\rangle \\ &\geq \frac{1}{L}\|\nabla f(X(t)) - \nabla f(x^\star)\|^2,\end{aligned}$$

where $\langle \eta'(t) - (-\nabla f(x^\star)), X(t) - x^\star\rangle \geq 0$ by monotonicity of $\partial g$. Then by (3.9) we obtain

$$\int_{t_0}^{+\infty} \|\nabla f(X(t)) - \nabla f(x^\star)\|^2 \, dt < +\infty \quad a.s.. \tag{3.12}$$

Let $\Omega_{\mathrm{HS}} \in \mathcal{F}$ be the event where (3.9) (and consequently (3.12)) is satisfied ( $\mathbb{P}(\Omega_{\mathrm{HS}}) = 1$). Let $\Omega_\eta \in \mathcal{F}$ be the event where $\eta'(t) \in \partial g(X(t))$ for almost all $T > t_0$ ($\mathbb{P}(\Omega_\eta) = 1$). Finally, let $\Omega_{\mathrm{conv}} \stackrel{\text{def}}{=} \widetilde{\Omega} \cap \Omega_{\mathrm{cont}} \cap \Omega_{\mathrm{HS}} \cap \Omega_M \cap \Omega_\eta$, hence $\mathbb{P}(\Omega_{\mathrm{conv}}) = 1$. Let also $\omega \in \Omega_{\mathrm{conv}} \subseteq \Omega_{\mathrm{HS}}$ arbitrary, then

$$\liminf_{t\to+\infty} \|\nabla f(X(\omega, t)) - \nabla f(x^\star)\| = 0.$$

If also

$$\limsup_{t\to+\infty} \|\nabla f(X(\omega, t)) - \nabla f(x^\star)\| = 0$$

then we conclude with the proof. Suppose by contradiction that there exists $\omega_0 \in \Omega_{\mathrm{conv}}$ such that

$$\limsup_{t\to+\infty} \|\nabla f(X(\omega_0, t)) - \nabla f(x^\star)\| > 0.$$

Then, by Lemma A.3, there exists $\delta(\omega_0) > 0$ satisfying

$$0 = \liminf_{t\to+\infty} \|\nabla f(X(\omega_0, t)) - \nabla f(x^\star)\| < \delta(\omega_0) < \limsup_{t\to+\infty} \|\nabla f(X(\omega_0, t)) - \nabla f(x^\star)\|,$$

and there exists $(t_k)_{k\in\mathbb{N}} \subset \mathbb{R}_+$ such that $\lim_{k\to+\infty} t_k = +\infty$,

$$\|\nabla f(X(\omega_0, t_k)) - \nabla f(x^\star)\| > \delta(\omega_0) \text{ and } t_{k+1} - t_k > 1, \quad \forall k \in \mathbb{N}.$$

Additionally, consider $\eta'(\omega_0, t) \in \partial g(X(\omega_0, t))$ for almost all $T > t_0$. Since $\sup_{t\geq t_0} \|X(\omega_0, t)\| < +\infty$, $\partial g$ is full domain, and the fact that $\partial g$ maps bounded sets onto bounded sets, we have that there exists $C_\eta(\omega_0) \geq 0$ such that $\|\eta'(\omega_0, t)\|^2 \leq C_\eta(\omega_0)$ for almost all $T > t_0$.

We allow ourselves the abuse of notation $X(t) \stackrel{\text{def}}{=} X(\omega_0, t), \eta'(t) \stackrel{\text{def}}{=} \eta'(\omega_0, t), C_\eta \stackrel{\text{def}}{=} C_\eta(\omega_0)$ and $\delta \stackrel{\text{def}}{=} \delta(\omega_0)$ during the rest of the proof from this point.

Let

- $C_0 \stackrel{\text{def}}{=} C_\eta + \|\nabla f(x^\star)\|^2$;
- $C_1 \stackrel{\text{def}}{=} \frac{(2C_0+1)^2-1}{C_0} > 0$;
- $\varepsilon \in \left]0, \min\{\frac{\delta^2}{4L^2}, C_1\}\right[$;
- and $C(\varepsilon) \stackrel{\text{def}}{=} \frac{\sqrt{C_0\varepsilon+1}-1}{4C_0} \in ]0, \frac{1}{2}]$.

Note that this choice entails that the intervals $([t_k, t_k + C(\varepsilon)])_{k\in\mathbb{N}}$ are disjoint. On the other hand, according to the convergence property of $M_t$ and the fact that $\|\nabla f(X(t)) - \nabla f(x^\star)\| \in \mathrm{L}^2([t_0, +\infty[)$, there exists $k' > 0$ such that for every $k \geq k'$

$$\sup_{t\geq t_k} \|N_t - N_{t_k}\|^2 < \frac{\varepsilon}{4} \text{ and } \int_{t_k}^{+\infty} \|\nabla f(X(t)) - \nabla f(x^\star)\|^2 dt < 1.$$

Also, we compute

$$\begin{aligned}\int_{t_k}^{t} \left\|\eta'(s) + \nabla f(X(s))\right\|^2 ds &\leq 2\int_{t_k}^{t} \|\nabla f(X(s)) - \nabla f(x^\star)\|^2 ds + 2\int_{t_k}^{t} \left\|\eta'(s) + \nabla f(x^\star)\right\|^2 ds \\ &\leq 2 + 4C_0(t - t_k).\end{aligned}$$

Furthermore, $C(\varepsilon)$ was chosen such that $C(\varepsilon) + 2C_0 C(\varepsilon)^2 \leq \frac{\varepsilon}{8}$. Besides for every $k \geq k'$, $t \in [t_k, t_k + C(\varepsilon)]$,

$$\begin{aligned}\|X(t) - X(t_k)\|^2 &\leq 2(t - t_k) \int_{t_k}^{t} \left\|\eta'(s) + \nabla f(X(s))\right\|^2 ds + 2\|N_t - N_{t_k}\|^2 \\ &\leq 4(t - t_k) + 8C_0(t - t_k)^2 + \frac{\varepsilon}{2} \leq \varepsilon.\end{aligned}$$

Since $\nabla f$ is $L$-Lipschitz and $L^2\varepsilon \leq \left(\frac{\delta}{2}\right)^2$ by assumption on $\varepsilon$, we have that for every $k \geq k'$ and $t \in [t_k, t_k + C(\varepsilon)]$

$$\|\nabla f(X(t)) - \nabla f(X(t_k))\|^2 \leq L^2 \|X(t) - X(t_k)\|^2 \leq \left(\frac{\delta}{2}\right)^2.$$

Therefore, for every $k \geq k'$, $t \in [t_k, t_k + C(\varepsilon)]$

$$\|\nabla f(X(t)) - \nabla f(x^\star)\| \geq \|\nabla f(X(t_k)) - \nabla f(x^\star)\| - \underbrace{\|\nabla f(X(t)) - \nabla f(X(t_k))\|}_{\leq \frac{\delta}{2}} \geq \frac{\delta}{2}.$$

Finally,

$$\begin{aligned}\int_{t_0}^{+\infty} \|\nabla f(X(s)) - \nabla f(x^\star)\|^2 ds &\geq \sum_{k \geq k'} \int_{t_k}^{t_k + C(\varepsilon)} \|\nabla f(X(s)) - \nabla f(x^\star)\|^2 ds \\ &\geq \sum_{k \geq k'} \frac{\delta^2 C(\varepsilon)}{4} = +\infty,\end{aligned}$$

which contradicts $\|\nabla f(X(\cdot)) - \nabla f(x^\star)\| \in \mathrm{L}^2([t_0, +\infty[)$. So, for every $\omega \in \Omega_{\text{conv}}$,

$$\begin{aligned}\limsup_{t \to +\infty} \|\nabla f(X(\omega, t)) - \nabla f(x^\star)\| &= \liminf_{t \to +\infty} \|\nabla f(X(\omega, t)) - \nabla f(x^\star)\| \\ &= \lim_{t \to +\infty} \|\nabla f(X(\omega, t)) - \nabla f(x^\star)\| = 0.\end{aligned}$$

On the other hand, since $F$ is convex, by (3.9), we obtain

$$\int_{t_0}^{+\infty} F(X(t)) - \min F \, dt < +\infty, \quad a.s.. \tag{3.13}$$

Since $\sup_{t \geq t_0} \|X(t)\| < +\infty$ a.s., and $(\partial F)$ maps bounded sets onto bounded sets (since $g$ is convex and continuous), we can show that there exists $\tilde{L} > 0$ such that

$$|F(X(t_1)) - F(X(t_2))| \leq \tilde{L}\|X(t_1) - X(t_2)\|, \quad \forall t_1, t_2 \geq t_0, a.s..$$

Using the same technique as before, we can conclude that $\lim_{t \to +\infty} F(X(t)) = \min F$ a.s..

(iv) Let $\omega \in \Omega_{\text{conv}}$ and $\widetilde{X}(\omega)$ be a weak sequential limit point of $X(\omega, t)$. Equivalently, there exists an increasing sequence $(t_k)_{k \in \mathbb{N}} \subset \mathbb{R}_+$ such that $\lim_{k \to +\infty} t_k = +\infty$ and

$$\operatorname*{w-lim}_{k \to +\infty} X(\omega, t_k) = \widetilde{X}(\omega).$$

Since $\lim_{t\to+\infty} F(X(\omega,t)) = \min F$ and the fact that $f$ is weakly lower semicontinuous (since it is convex and continuous), we obtain directly that $\widetilde{X}(\omega) \in \mathcal{S}_F$. Finally, by Opial's Lemma (see [25]) we conclude that there exists $X^\star(\omega) \in \mathcal{S}_F$ such that

$$\underset{t\to+\infty}{\text{w-lim}}\, X(\omega,t) = X^\star(\omega).$$

In other words, since $\omega \in \Omega_{\text{conv}}$ was arbitrary, there exists an $\mathcal{S}_F$-valued random variable $X^\star$ such that $\text{w-lim}_{t\to+\infty} X(t) = X^\star$ a.s..

$\square$

### 3.3 Convergence rates of the objective

Our first result, stated below, summarizes the global convergence rates in expectation satisfied by the trajectories of (SDI).

**Theorem 3.8.** *Consider the dynamic (SDI) where $F = f + g$ and $\sigma$ satisfy the assumptions ($\mathrm{H}_0$) and (H), furthermore assume that $\partial g$ satisfies ($H_\lambda$). Additionally, $X_0 \in \mathrm{L}^2(\Omega;\mathbb{H})$ and is $\mathcal{F}_0$-measurable. The following statements are satisfied by the unique solution trajectory $X \in S^2_{\mathbb{H}}[t_0]$ of (SDI):*

*(i) Let $\overline{F\circ X}(t) \overset{\text{def}}{=} t^{-1}\int_{t_0}^t F(X(s))ds$ and $\overline{X}(t) = t^{-1}\int_{t_0}^t X(s)ds$. Then*

$$\mathbb{E}\left(F(\overline{X}(t)) - \min F\right) \le \mathbb{E}\left(\overline{F\circ X}(t) - \min F\right) \le \frac{\mathbb{E}\left(\text{dist}(X_0,\mathcal{S}_F)^2\right)}{2t} + \frac{\sigma_*^2}{2}, \quad \forall t > t_0. \tag{3.14}$$

*Besides, if $\sigma_\infty$ is $\mathrm{L}^2([t_0,+\infty[)$, then*

$$\mathbb{E}\left(F(\overline{X}(t)) - \min F\right) \le \mathbb{E}\left(\overline{F\circ X}(t) - \min F\right) = \mathcal{O}\left(\frac{1}{t}\right). \tag{3.15}$$

*(ii) Moreover, if $F \in \Gamma_\mu(\mathbb{H})$ with $\mu > 0$, then $\mathcal{S}_F = \{x^\star\}$ and*

$$\mathbb{E}\left(\|X(t) - x^\star\|^2\right) \le \mathbb{E}\left(\|X_0 - x^\star\|^2\right) e^{-\mu t} + \frac{\sigma_*^2}{\mu}, \quad \forall t > t_0. \tag{3.16}$$

*Besides, if $\sigma_\infty$ is non-increasing and vanishes at infinity, then:*

$$\mathbb{E}\left(\|X(t) - x^\star\|^2\right) \le \mathbb{E}\left(\|X_0 - x^\star\|^2\right) e^{-\mu t} + \frac{\sigma_*^2}{\mu} e^{\frac{\mu t_0}{2}} e^{-\frac{\mu t}{2}} + \sigma_\infty^2\left(\frac{t_0+t}{2}\right), \quad \forall t > t_0. \tag{3.17}$$

**Proof.** Using that $F(x) - \min F \le \langle y, x - x^\star\rangle$ for every $y \in \partial F(x), x^\star \in \mathcal{S}_F$, and Itô's formula with the anchor function $\phi(x) = \frac{\|x-x^\star\|^2}{2}$ (for $x^\star \in \mathcal{S}_F$), the proof is analogous to [1, Theorem 3.2]. $\square$

## 4 Tikhonov regularization: Convergence properties for convex functions

It is important to provide insight into the technique of Tikhonov regularization. This allows us to pass from the almost sure weak convergence towards the set of minimizers of the trajectory generated by ($\mathrm{SDI}_0$) to achieving almost sure strong convergence of the trajectory generated by ($\mathrm{SDI-TA}$), not only towards the set of minimizers but to the minimal norm solution. The price to pay in order to achieve this is the proper tuning of the Tikhonov parameter that depends on a local constant that could be hard to compute, besides that, we obtain slower convergence rates of the objective, passing from $\mathcal{O}(t^{-1})$ to $\mathcal{O}(t^{-r} + R(t))$, where $r < 1$ and $R(t) \to 0$ (defined below in (4.12)).

### 4.1 Almost sure convergence of the trajectory to the minimal norm solution

Our second main result establish almost sure convergence of $X(t)$ to $x^\star = \mathrm{P}_{\mathcal{S}_F}(0)$ as $t \to +\infty$. It is based on a subtle tuning of the Tikhonov parameter $\varepsilon(t)$ formulated as conditions $(T_1)$, $(T_2)$, and $(T_3)$ below. We know that $\|x^\star\|^2 - \|x_{\varepsilon(t)}\|^2$ tends to zero as $t \to +\infty$. We shall see that the conditions $(T_1)$, $(T_2)$, and $(T_3)$ are compatible for tame functions (*i.e.* which satisfy a Kurdyka-Lojasiewicz property).

**Theorem 4.1.** *Consider the dynamic* (SDI − TA) *where $F = f + g$ and $\sigma$ satisfy the assumptions* ($\mathrm{H}_0$) *and* (H)*, respectively, furthermore assume that $\partial g$ satisfy* ($H_\lambda$)*. Let $\nu \geq 2$, and its initial data $X_0 \in \mathrm{L}^\nu(\Omega; \mathbb{H})$. Then, there exists a unique solution $X \in S_{\mathbb{H}}^\nu[t_0]$ of* (SDI − TA)*. Let $x^\star \stackrel{\text{def}}{=} \mathrm{P}_{\mathcal{S}_F}(0)$ be the minimum norm solution, and for $\varepsilon > 0$ let $x_\varepsilon$ be the unique minimizer of $F_\varepsilon(x) \stackrel{\text{def}}{=} F(x) + \frac{\varepsilon}{2}\|x\|^2$. Suppose that $\sigma_\infty \in \mathrm{L}^2(\mathbb{R}_+)$, and that $\varepsilon : [t_0, +\infty[ \to \mathbb{R}_+$ satisfies the conditions:*

$(T_1)$ $\varepsilon(t) \to 0$ *as* $t \to +\infty$;

$(T_2)$ $\displaystyle\int_{t_0}^{+\infty} \varepsilon(t)dt = +\infty$;

$(T_3)$ $\displaystyle\int_{t_0}^{+\infty} \varepsilon(t)\left(\|x^\star\|^2 - \|x_{\varepsilon(t)}\|^2\right) dt < +\infty.$

*Then we have*

*(i)* $\displaystyle\int_{t_0}^{+\infty} \varepsilon(t)\mathbb{E}[\|X(t) - x^\star\|^2]dt < +\infty.$

*(ii)* $\lim_{t\to+\infty} \|X(t) - x^\star\|$ *exists a.s. and* $\sup_{t\geq t_0} \|X(t)\| < +\infty$ *a.s.*.

*(iii)* $\displaystyle\int_{t_0}^{+\infty} \varepsilon(t)\|X(t) - x^\star\|^2 dt < +\infty$ *a.s.*.

*(iv)* $\text{s-}\lim_{t\to+\infty} X(t) = x^\star$ *a.s.*

**Proof.** The existence and uniqueness of a solution $X \in S_{\mathbb{H}}^\nu[t_0]$ follow directly from the fact that the conditions of Theorem 3.3 are satisfied under ($\mathrm{H}_0$) and (H). The only subtlety to check is that $\sup_{t\geq t_0} |\varepsilon(t)| < +\infty$, but this can be assumed without loss of generality since $\varepsilon(t) \to 0$ as $t \to +\infty$ (it might be necessary a re-definition of $t_0$).

Our stochastic dynamic (SDI − TA) can be written equivalently as follows

$$\begin{cases} dX(t) \in -\partial F_{\varepsilon(t)}(X(t))dt + \sigma(t, X(t))dW(t), & t \geq t_0; \\ X(t_0) = X_0, \end{cases} \tag{SDIT}$$

(i) Let us define the anchor function $\phi(x) = \frac{\|x-x^\star\|^2}{2}$. Since $\partial g$ satisfy ($H_\lambda$), there exists a stochastic process $\tilde{\eta} : \Omega \times [t_0, +\infty[ \to \mathbb{H}$ such that $\tilde{\eta}(t) \in \partial F_{\varepsilon(t)}(X(t))$ a.s. for almost all $t \geq t_0$. Using Itô's formula we obtain

$$\begin{aligned} \phi(X(t)) = \underbrace{\frac{\|X_0 - x^\star\|^2}{2}}_{\xi} + \underbrace{\frac{1}{2}\int_{t_0}^t \mathrm{tr}\left(\Sigma(s, X(s))\right) ds}_{A_t} - \underbrace{\int_{t_0}^t \langle \tilde{\eta}(s), X(s) - x^\star \rangle\, ds}_{U_t} \\ + \underbrace{\int_{t_0}^t \langle \sigma^\star(s, X(s))\left(X(s) - x^\star\right), dW(s)\rangle}_{M_t}. \end{aligned} \tag{4.1}$$

Since $X \in S^2_{\mathbb{H}}[t_0]$ by Proposition 3.5, we have for every $T > t_0$, that

$$\mathbb{E}\left(\int_{t_0}^{T} \|\sigma^\star(s, X(s))\,(X(s) - x^\star)\|^2\, ds\right) \le \mathbb{E}\left(\sup_{t \in [t_0, T]} \|X(t) - x^\star\|^2\right) \int_{t_0}^{+\infty} \sigma_\infty^2(s) ds < +\infty.$$

Therefore $M_t$ is a square-integrable continuous martingale. It is also a continuous local martingale, which implies that $\mathbb{E}(M_t) = 0$.
Let us now take the expectation of (4.1). Using that

$$0 \le \operatorname{tr}(\Sigma(s, X(s))) \le \sigma_\infty^2(s)$$

and (2.3) that we recall below

$$\langle y(t), X(t) - x^\star\rangle \ge \varepsilon(t)\phi(X(t)) + \frac{\varepsilon(t)}{2}\left(\|x_{\varepsilon(t)}\|^2 - \|x^\star\|^2\right), \tag{4.2}$$

where $y : \Omega \times [t_0, +\infty[\to \mathbb{H}$ is such that $y(t) \in \partial F_{\varepsilon(t)}(X(t))$ a.s., we obtain that

$$\begin{aligned}
&\mathbb{E}\left(\phi(X(t))\right) + \int_{t_0}^{t} \varepsilon(s)\mathbb{E}\left(\phi(X(s))\right) ds \\
&\le \mathbb{E}\left(\frac{\|X_0 - x^\star\|^2}{2}\right) + \frac{1}{2}\int_{t_0}^{t} \sigma_\infty^2(s) ds + \frac{1}{2}\int_{t_0}^{t} \varepsilon(s)\left(\|x^\star\|^2 - \|x_{\varepsilon(s)}\|^2\right) ds.
\end{aligned}$$

According to our assumptions, we can write briefly the above relation as

$$\mathbb{E}\left(\phi(X(t))\right) + \int_{t_0}^{t} \varepsilon(s)\mathbb{E}\left(\phi(X(s))\right) ds \le g(t), \tag{4.3}$$

with $g$ a nonnegative function defined by

$$g(t) := \mathbb{E}\left(\frac{\|X_0 - x^\star\|^2}{2}\right) + \frac{1}{2}\int_{t_0}^{t} \sigma_\infty^2(s) ds + \frac{1}{2}\int_{t_0}^{t} \varepsilon(s)\left(\|x^\star\|^2 - \|x_{\varepsilon(s)}\|^2\right) ds$$

which satisfies $\lim_{t \to +\infty} g(t) = g_\infty < +\infty$.

Let us integrate the above relation (4.3). We set

$$\theta(t) := \int_{t_0}^{t} \mathbb{E}\left(\phi(X(s))\right) ds.$$

We have $\dot{\theta}(t) = \mathbb{E}\left(\phi(X(t))\right)$ and (4.3) is written equivalently as

$$\dot{\theta}(t) + \int_{t_0}^{t} \varepsilon(s)\dot{\theta}(s) ds \le g(t). \tag{4.4}$$

Equivalently

$$\frac{1}{\varepsilon(t)}\frac{d}{dt}\int_{t_0}^{t} \varepsilon(s)\dot{\theta}(s) ds + \int_{t_0}^{t} \varepsilon(s)\dot{\theta}(s) ds \le g(t), \tag{4.5}$$

that is

$$\frac{d}{dt}\int_{t_0}^t \varepsilon(s)\dot\theta(s)ds + \varepsilon(t)\int_{t_0}^t \varepsilon(s)\dot\theta(s)ds \le \varepsilon(t)g(t). \tag{4.6}$$

With $m(t) := \exp\int_{t_0}^t \varepsilon(s)ds$ we get

$$\frac{d}{dt}\left(m(t)\int_{t_0}^t \varepsilon(s)\dot\theta(s)ds\right) \le \varepsilon(t)m(t)g(t). \tag{4.7}$$

After integration we get

$$\int_{t_0}^t \varepsilon(s)\dot\theta(s)ds \le \frac{1}{m(t)}\int_{t_0}^t m'(s)g(s)ds. \tag{4.8}$$

Since $g$ is bounded by assumption $(T_2)$, we get

$$\sup_{t\ge t_0}\mathbb{E}\Big[\int_{t_0}^t \varepsilon(s)\|X(s)-x^\star\|^2\Big]ds < +\infty.$$

Equivalently

$$\int_{t_0}^{+\infty}\mathbb{E}\Big[\|X(t)-x^\star\|^2\Big]\varepsilon(t)dt < +\infty.$$

The assumption $(T_2)$ guarantees that the above inequality forces $\mathbb{E}\Big[\|X(t)-x^\star\|^2\Big]$ to tend to zero.

(ii) Consider (4.1), we define

$$\tilde A_t \overset{\text{def}}{=} A_t + \int_{t_0}^t \frac{\varepsilon(s)}{2}(\|x^\star\|^2 - \|x_{\varepsilon(s)}\|^2)ds, \quad and \quad \tilde U_t \overset{\text{def}}{=} U_t + \int_{t_0}^t \frac{\varepsilon(s)}{2}(\|x^\star\|^2 - \|x_{\varepsilon(s)}\|^2)ds.$$

By (4.2) we have that $\tilde U_t \ge \int_{t_0}^t \varepsilon(s)\phi(X(s))ds \ge 0$. We can rewrite (4.1) as

$$\phi(X(t)) = \xi + \tilde A_t - \tilde U_t + M_t.$$

Since $\sigma_\infty \in \mathrm{L}^2([t_0,+\infty[)$ and $(T_3)$, then $\lim_{t\to+\infty}\tilde A_t < +\infty$. Let us observe that, since $X \in S^2_{\mathbb{H}}[t_0]$ by Proposition 3.5, we have for every $T > t_0$ that

$$\mathbb{E}\left(\int_{t_0}^T \|\sigma^\star(s,X(s))\,(X(s)-x^\star)\|^2\,ds\right) \le \mathbb{E}\left(\sup_{t\in[t_0,T]}\|X(t)-x^\star\|^2\right)\int_{t_0}^{+\infty}\sigma_\infty^2(s)ds < +\infty.$$

Therefore $M_t$ is a square-integrable continuous martingale. It is also a continuous local martingale (see [24, Theorem 1.3.3]), which implies that $\mathbb{E}(M_t) = 0$.

By Theorem A.7, we get that $\lim_{t\to+\infty}\|X(t)-x^\star\|$ exists a.s. and that $\lim_{t\to+\infty}\tilde U_t < +\infty$ a.s..

(iii) Using the lower bound we had on $\tilde U_t$, we obtain

$$\int_{t_0}^{+\infty}\varepsilon(t)\,\|X(t)-x^\star\|^2\,dt < +\infty.$$

(iv) By the previous item, $(T_2)$, and Lemma A.1 we conclude that $\lim_{t\to+\infty}X(t) = x^\star$ a.s..

This completes the proof. $\square$

### 4.2 Practical situations

We will consider situations where the three conditions $(T_1)$, $(T_2)$ and $(T_3)$ are satisfied simultaneously. These are properties of the viscosity curve that we will now study. The difficulty comes from $(T_2)$ and $(T_3)$ which are a priori not compatible. Indeed, $(T_2)$ requires the parameter $\varepsilon(t)$ to converge slowly towards zero for the Tikhonov regularization to be effective. On the other hand in $(T_3)$ the parameter $\varepsilon(t)$ must converge sufficiently quickly towards zero so that the term $\left(\|x^\star\|^2 - \|x_{\varepsilon(t)}\|^2\right)$ converges to zero fairly quickly, and thus corrects the infinite value of the integral of $\varepsilon(t)$.

#### 4.2.1 Łojasiewicz property

Our first objective is to evaluate the rate of convergence towards zero of $\left(\|x^\star\|^2 - \|x_\varepsilon\|^2\right)$ as $\varepsilon \to 0$. Using the differentiability properties of the viscosity curve is not a good idea, because the viscosity curve can be of infinite length in the case of a general differentiable convex function, see [26]. To overcome this difficulty, we assume that $F = f + g$ satisfies the Łojasiewicz property. This basic property has its roots in algebraic geometry, and it essentially describes a relationship between the objective value and its gradient (or subgradient).

**Definition 4.2 (Łojasiewicz inequality).** Let $f : \mathbb{H} \to \mathbb{R}$ be a convex function with $\mathcal{S} \neq \emptyset$ and $q \in [0, 1[$. $f$ satisfies the Łojasiewicz inequality on $\mathcal{S}$ with exponent $q$ if there exists $r > \min f$ and $\mu > 0$ such that

$$\mu(f(x) - \min f)^q \leq \left\|\partial^0 f(x)\right\|, \quad \forall x \in [\min f < f < r], \tag{4.9}$$

and we will write $f \in Ł^q(\mathcal{S})$.

Error bounds have also been successfully applied to various branches of optimization, and in particular to complexity analysis. Of particular interest in our setting is the Hölderian error bound.

**Definition 4.3 (Hölderian error bound).** Let $f : \mathbb{H} \to \mathbb{R}$ be a proper function such that $\mathcal{S} \neq \emptyset$. Then $f$ satisfies a Hölderian (or power-type) error bound inequality on $\mathcal{S}$ with exponent $p \geq 1$, if there exists $\gamma > 0$ and $r > \min f$ such that:

$$f(x) - \min f \geq \gamma \mathrm{dist}(x, \mathcal{S})^p, \quad \forall x \in [\min f \leq f \leq r], \tag{4.10}$$

and we will write $f \in \mathrm{EB}^p(\mathcal{S})$,

A deep result due to Łojasiewicz states that for arbitrary continuous semi-algebraic functions, the H"olderian error bound inequality holds on any compact set, and the Łojasiewicz inequality holds at each point. In fact, for convex functions, the Łojasiewicz property and Hölderian error bound are actually equivalent.

**Proposition 4.4.** *Assume that $f \in \Gamma_0(\mathbb{H})$ with $\mathcal{S} \neq \emptyset$. Let $q \in [0, 1[$, $p \stackrel{\text{def}}{=} \frac{1}{1-q} \geq 1$ and $r > \min f$. Then $f$ verifies the Łojasiewicz inequality with exponent $q$ (see (4.9)) at $[\min f < f < r]$ if and only if the Hölderian error bound with exponent $p$ (see (4.10)) holds on $[\min f < f < r]$.*

#### 4.2.2 Quantitative stability of variational systems

Let us recall the following two results that have been obtained in the following papers:

[AW1] H. Attouch, R. Wets, Quantitative stability of variational systems: I, The epigraphical distance, *Transactions of the American Mathematical Society*, 328 (1991), No. 2, pp. 695–729.

[AMR] H. Attouch, A. Moudafi, H. Riahi, Quantitative stability analysis for maximal monotone operators and semi-groups of contractions, *Nonlinear Analysis, TMA*, 21 (1993), No. 9, pp. 697–723.

Consider $\mathcal{H}$ a Hilbert space.

**Proposition 4.5.** *Let $A, B : \mathcal{H} \rightrightarrows \mathcal{H}$ be two maximal monotone operators, then*

$$\|(I+A)^{-1}(0) - \|(I+B)^{-1}(0)\| \leq 3haus_{\|(I+A)^{-1}(0)\|}(A,B).$$

**Theorem 4.6.** *Let $f$ and $g$ be convex lower semicontinuous proper functions from $\mathcal{H}$ into $\mathbb{R}$. To any $\rho >$ $\max[\mathrm{dist}(0, \mathrm{epi}(f)), \mathrm{dist}(0, \mathrm{epi}(g))]$ there correspond some constants $\kappa$ and $\rho_0$ (that depend on $\rho$) such that*

$$haus_\rho(\partial f, \partial g) \leq \kappa[haus_{\rho_0}(f,g)]^{\frac{1}{2}}.$$

The second abstract result is the equivalence of the uniform structure on the class of subdifferentials of convex lsc. functions between the bounded Hausdorff distance and the uniform convergence on bounded sets of resolvents.

The following Proposition is new and is a consequence of the previous two results, since this is not obvious, we are going to present the whole proof.

**Proposition 4.7.** *Let $f \in \Gamma_0(\mathbb{H})$ be a function such that $\mathcal{S} \neq \emptyset$, and that $f \in \mathrm{EB}^p(\mathcal{S})$. Let also $x^\star = \mathrm{P}_\mathcal{S}(0)$ and for $\varepsilon > 0$, let $x_\varepsilon$ be the unique minimizer of $f_\varepsilon(x) = f(x) + \frac{\varepsilon}{2}\|x\|^2$. Then there exists $C_0, \varepsilon^\star > 0$ such that*

$$\|x_\varepsilon - x^\star\| \leq C_0 \varepsilon^{\frac{1}{2p}}, \quad \forall \varepsilon \in ]0, \varepsilon^\star].$$

*Consequently, there exists $C > 0$ such that*

$$\|x^\star\|^2 - \|x_\varepsilon\|^2 \leq C\varepsilon^{\frac{1}{2p}}, \quad \forall \varepsilon \in ]0, \varepsilon^\star].$$

**Proof.** We have that

$$x_\varepsilon + \frac{1}{\varepsilon}\partial f(x_\varepsilon) \ni 0,$$

that is

$$x_\varepsilon = (I + \partial\varphi_\varepsilon)^{-1}(0)$$

where

$$\varphi_\varepsilon \overset{\text{def}}{=} \frac{1}{\varepsilon}(f - \min f).$$

We have that $\varphi_\varepsilon$ increases to $\delta_\mathcal{S}$ as $\varepsilon$ decreases to zero, and

$$x^\star = \mathrm{P}_\mathcal{S}(0) = (I + \partial\delta_\mathcal{S})^{-1}(0),$$

where $\delta_\mathcal{S}$ is the characteristic function $(0 - +\infty)$ of $\mathcal{S}$. So

$$\|x_\varepsilon - x^\star\| = \|(I + \partial\varphi_\varepsilon)^{-1}(0) - (I + \partial\delta_\mathcal{S})^{-1}(0)\|.$$

By Theorem 4.5 with $A = \partial\varphi_\varepsilon$, and $B = \partial\delta_\mathcal{S}$, we have that

$$\|x_\varepsilon - x^\star\| \leq 3\mathrm{haus}_\rho(\partial\varphi_\varepsilon, \partial\delta_\mathcal{S}),$$

for $\rho > \|x^\star\|$. Now, since $\max[\mathrm{dist}(0, \mathrm{epi}(\varphi_\varepsilon)), \mathrm{dist}(0, \mathrm{epi}(\delta_{\mathcal{S}}))] \leq \|x^\star\|$, let $\rho > \|x^\star\|$, by Proposition 4.6 we obtain that there exists $\kappa, \rho_0 > 0$ constants (depending on $\rho$) such that

$$\|x_\varepsilon - x^\star\| \leq 3\kappa[\mathrm{haus}_{\rho_0}(\varphi_\varepsilon, \delta_{\mathcal{S}})]^{\frac{1}{2}}.$$

To complete our proof we just need to bound

$$\mathrm{haus}_{\rho_0}(\varphi_\varepsilon, \delta_{\mathcal{S}}).$$

To that end, we use the $\mathrm{EB}^p(\mathcal{S})$ assumption. Since $f \in \mathrm{EB}^p(\mathcal{S})$, there exists $\gamma > 0, r > \min f$ such that

$$f(x) - \min f \geq \gamma \mathrm{dist}(x, \mathcal{S})^p, \quad \forall x \in [\min f \leq f \leq r].$$

Since $\delta_{\mathcal{S}} \geq \varphi_\varepsilon$ we just need to compute the bounded Hausdorff excess of the epigraph of $\varphi_\varepsilon$ over $\mathcal{S} \times \{+\infty\}$. We have

$$\delta_{\mathcal{S}}(x) \geq \varphi_\varepsilon(x) \geq \frac{\gamma}{\varepsilon}\mathrm{dist}(x, \mathcal{S})^p, \quad \forall x \in [\min f \leq f \leq r].$$

By definition

$$\mathrm{haus}_{\rho_0}(\varphi_\varepsilon, \delta_{\mathcal{S}}) = \sup_{(x_1,r_1)\in\mathrm{epi}(\varphi_\varepsilon)\cap\rho_0\mathbb{B}} \inf_{(x_2,r_2)\in\mathrm{epi}(\delta_{\mathcal{S}})} \max[\mathrm{dist}(x_1, x_2), \mathrm{dist}(r_1, r_2)],$$

where $\mathbb{B}$ is the unit ball of $\mathbb{H} \times \mathbb{R}$. Besides,

$$\sup_{(x_1,r_1)\in\mathrm{epi}(\varphi_\varepsilon)\cap\rho_0\mathbb{B}} \inf_{(x_2,r_2)\in\mathrm{epi}(\delta_{\mathcal{S}})} \max[\mathrm{dist}(x_1, x_2), \mathrm{dist}(r_1, r_2)] \underset{r_2=r_1;x_2\in\mathcal{S}}{\leq} \sup_{(x_1,r_1)\in\mathrm{epi}(\varphi_\varepsilon)\cap\rho_0\mathbb{B}} \mathrm{dist}(x_1, \mathcal{S}).$$

Now let $\varepsilon_0 \overset{\text{def}}{=} \frac{r-\min f}{\rho_0} > 0$ and consider $\varepsilon \in ]0, \varepsilon_0]$, then

$$\sup_{(x_1,r_1)\in\mathrm{epi}(\varphi_\varepsilon)\cap\rho_0\mathbb{B}} \mathrm{dist}(x_1, \mathcal{S}) \leq \sup_{x_1\in[f\leq\rho_0\varepsilon+\min f]} \mathrm{dist}(x_1, \mathcal{S}) \leq \left(\frac{\rho_0}{\gamma}\right)^{\frac{1}{p}} \varepsilon^{\frac{1}{p}},$$

where we have used that $\rho_0\varepsilon + \min f \leq r$ and the hypothesis $\mathrm{EB}^p(\mathcal{S})$ in the last inequality.

Let $C_0 \overset{\text{def}}{=} 3\kappa \left(\frac{\rho_0}{\gamma}\right)^{\frac{1}{2p}}$, then for $\varepsilon \in ]0, \varepsilon_0]$,

$$\|x_\varepsilon - x^\star\| \leq C_0\varepsilon^{\frac{1}{2p}}.$$

On the other hand, since $\lim_{\varepsilon\to0^+} x_\varepsilon = x^\star$, then there exists $\varepsilon_1 > 0$ such that $\|x_\varepsilon\| \leq 1 + \|x^\star\|$, for every $\varepsilon \leq \varepsilon_1$. Let $C_1 \overset{\text{def}}{=} 1 + 2\|x^\star\|$, from

$$\|x^\star\|^2 - \|x_\varepsilon\|^2 \leq C_1\|x_\varepsilon - x^\star\|, \quad \forall\varepsilon \in ]0, \varepsilon_1].$$

And letting $\varepsilon^\star \overset{\text{def}}{=} \min \varepsilon_0, \varepsilon_1$, and $C \overset{\text{def}}{=} C_0C_1$, then we get

$$\|x^\star\|^2 - \|x_\varepsilon\|^2 \leq C\varepsilon^{\frac{1}{2p}}, \forall\varepsilon \in ]0, \varepsilon^\star].$$

□

**Theorem 4.8.** *Consider the setting of Theorem 4.1 and suppose that $F = f + g \in \mathrm{EB}^p(\mathcal{S}_F)$. Then taking the Tikhonov parameter $\varepsilon(t) = \frac{1}{t^r}$ with*

$$1 \geq r > \frac{2p}{2p+1},$$

*then the three conditions $(T_1)$, $(T_2)$, and $(T_3)$ of Theorem 4.1 are satisfied simultaneously. In particular, the solution $X \in S^{\nu}_{\mathbb{H}}[t_0]$ of (SDI − TA) is unique and we get almost sure (strong) convergence of $X(t)$ to the minimal norm solution named $x^\star = \mathrm{P}_{\mathcal{S}_F}(0)$.*

**Proof.** It is direct to check $(T_1)$ and $(T_2)$. In order to check $(T_3)$, let $\varepsilon^\star > 0$ from Proposition 4.7 and $T^\star = \max[t_0, (\frac{1}{\varepsilon^\star})^{\frac{1}{r}}]$, then we have

$$\|x^\star\|^2 - \|x_{\varepsilon(t)}\|^2 \leq C\frac{1}{t^{\frac{r}{2p}}}, \quad \forall t \geq T^\star.$$

So

$$\int_{t_0}^{+\infty} \frac{\|x^\star\|^2 - \|x_{\varepsilon(t)}\|^2}{t^r} dt = \underbrace{\int_{t_0}^{T^\star} \frac{\|x^\star\|^2 - \|x_{\varepsilon(t)}\|^2}{t^r} dt}_{I_1} + \underbrace{\int_{T^\star}^{+\infty} \frac{\|x^\star\|^2 - \|x_{\varepsilon(t)}\|^2}{t^r} dt}_{I_2}.$$

Is clear that $I_1$ is bounded (by $T^\star t_0^{-r}\|x^\star\|^2$ for instance), hence $(T_3)$ holds under the condition that

$$\int_{T^\star}^{+\infty} \frac{1}{t^r}\frac{C}{t^{\frac{r}{2p}}} dt < +\infty$$

which is true when $r + \frac{r}{2p} > 1$, which gives the condition $1 \geq r > \frac{2p}{2p+1}$. □

### 4.3 Convergence rates of the objective in the smooth case

We are going to show global convergence rates in expectation in the smooth convex case, in order to do that, it is worth citing a result from [10], where they deal with the deterministic case.

Take $\varepsilon(t) = \dfrac{1}{t^r}$, $0 < r < 1$, $t_0 > 0$. The convergence rate of the values and the strong convergence to the minimum norm solution is described below, see Attouch, Chbani, Riahi [10, Theorem 5].

**Theorem 4.9.** *Take $\varepsilon(t) = \dfrac{1}{t^r}$ and $0 < r < 1$. Let us consider (DI − TA) in the case where $g \equiv 0$, i.e.*

$$\dot{x}(t) + \nabla f(x(t)) + \frac{1}{t^r}x(t) = 0. \tag{4.11}$$

*Let $x : [t_0, +\infty[ \to \mathcal{H}$ be a solution trajectory of (DI − TA). For $\varepsilon > 0$ define $f_\varepsilon(x) \stackrel{\text{def}}{=} f(x) + \frac{\varepsilon}{2}\|x\|^2$, let $x_\varepsilon$ be the unique minimizer of $f_\varepsilon$, and consider the Lyapunov function*

$$E(t) \stackrel{\text{def}}{=} f_{\varepsilon(t)}(x(t)) - f_{\varepsilon(t)}(x_{\varepsilon(t)}) + \frac{\varepsilon(t)}{2}\|x(t) - x_{\varepsilon(t)}\|^2.$$

*Then, we have*

*(i)* $E(t) = \mathcal{O}\left(\dfrac{1}{t}\right)$ *as* $t \to +\infty$;

*(ii)* $f(x(t)) - \min(f) = \mathcal{O}\left(\frac{1}{t^r}\right)$ *as* $t \to +\infty$;

*(iii)* $\|x(t) - x_{\varepsilon(t)}\|^2 = \mathcal{O}\left(\frac{1}{t^{1-r}}\right)$ *as* $t \to +\infty$.

*In addition, we have strong convergence of $x(t)$ to the minimum norm solution, named $x^\star = \mathrm{P}_{\mathcal{S}}(0)$. Moreover, if $f \in \mathrm{EB}^p(\mathcal{S})$*

*(iv)* $$\|x(t) - x^\star\|^2 = \begin{cases} \mathcal{O}\left(\frac{1}{t^{\frac{r}{p}}}\right), & \text{if } r \in \left]0, \frac{p}{p+1}\right[; \\ \mathcal{O}\left(\frac{1}{t^{1-r}}\right), & \text{if } r \in \left[\frac{p}{p+1}, 1\right[ \end{cases} \quad \text{as } t \to +\infty.$$

**Remark 4.10.** The last item of this already known Theorem is new and direct from our Proposition 4.7.

Now we have the necessary tools in order to show our first result in this sense, this one summarizes the global convergence rates in expectation satisfied by the trajectories of (SDI − TA) in the case where $g \equiv 0$.

**Theorem 4.11.** *Let $\nu \geq 2$, $f \in \Gamma_0(\mathbb{H}) \cap C_L^2(\mathbb{H})$ such that $\mathcal{S} \stackrel{\text{def}}{=} \mathrm{argmin}(f)$ is nonempty, and also $f \in \mathrm{EB}^p(\mathcal{S})$, $\sigma$ satisfying (H), and $\sigma_\infty \in \mathrm{L}^2([t_0, +\infty[)$ and is non-increasing. Let us consider $\varepsilon(t) = \frac{1}{t^r}$ where $0 < r < 1$, then we evaluate (SDI − TA) in the case where $g \equiv 0$, and with initial data $X_0 \in \mathrm{L}^\nu(\Omega; \mathbb{H})$, i.e.*

$$\begin{cases} dX(t) = -\nabla f(X(t))dt - \frac{1}{t^r}X(t)dt + \sigma(t, X(t))dW(t), & t \geq t_0; \\ X(t_0) = X_0. \end{cases} \tag{SDE − TA}$$

*For $\varepsilon > 0$, let us define $f_\varepsilon(x) \stackrel{\text{def}}{=} f(x) + \frac{\varepsilon}{2}\|x\|^2$, $x_\varepsilon$ be the unique minimizer of $f_\varepsilon$, consider the Lyapunuov function*

$$E(t, x) \stackrel{\text{def}}{=} f_{\varepsilon(t)}(x) - f_{\varepsilon(t)}(x_{\varepsilon(t)}) + \frac{\varepsilon(t)}{2}\|x - x_{\varepsilon(t)}\|^2,$$

*and for $t_1 > t_0$,*

$$R(t) \stackrel{\text{def}}{=} e^{-\frac{t^{1-r}}{1-r}} \int_{t_1}^{t} e^{\frac{s^{1-r}}{1-r}} \sigma_\infty^2(s) ds. \tag{4.12}$$

*Consider $x^\star \stackrel{\text{def}}{=} \mathrm{P}_{\mathcal{S}}(0)$. Then, the solution trajectory $X \in S_{\mathbb{H}}^\nu[t_0]$ is unique, and we have that:*

*(i)* $R(t) \to 0$ *as* $t \to +\infty$;

*(ii) Furthermore, we can obtain a convergence rate for $R$,*

$$R(t) = \mathcal{O}\left(\exp(-t^r(1 - 2^{-r})) + t^r \sigma_\infty^2\left(\frac{t_1 + t}{2}\right)\right).$$

*Moreover, if $\sigma_\infty^2(t) = \mathcal{O}(t^{-\alpha})$ for $\alpha > 1$, then $R(t) = \mathcal{O}(t^{r-\alpha})$.*

*(iii)* $\mathbb{E}[E(t, X(t))] = \mathcal{O}\left(\frac{1}{t} + R(t)\right)$

*(iv)* $\mathbb{E}[f(X(t)) - \min(f)] = \mathcal{O}\left(\frac{1}{t^r} + R(t)\right)$. *Moreover if $\sigma_\infty^2(t) = \mathcal{O}(t^{-\alpha})$ for $\alpha > 1$, then*

$$\mathbb{E}[f(X(t)) - \min(f)] = \begin{cases} \mathcal{O}\left(\frac{1}{t^{\alpha-r}}\right), & \text{if } \alpha \in ]1, 2r[; \\ \mathcal{O}\left(\frac{1}{t^r}\right), & \text{if } \alpha \geq 2r; \end{cases}$$

*(v)* $\mathbb{E}[\|X(t)-x_{\varepsilon(t)}\|^2]=\mathcal{O}\left(\frac{1}{t^{1-r}}+t^rR(t)\right)$, *which goes to* $0$ *as* $t\to+\infty$ *if* $r\in]0,\frac{1}{2}]$. *Moreover, if* $\sigma_\infty^2(t)=\mathcal{O}(t^{-\alpha})$ *for* $\alpha>\max\{2r,1\}$, *then*

$$\mathbb{E}[\|X(t)-x_{\varepsilon(t)}\|^2]=\begin{cases}\mathcal{O}\left(\frac{1}{t^{\alpha-2r}}\right), & \text{if } \alpha\in]\max\{1,2r\},r+1[;\\ \mathcal{O}\left(\frac{1}{t^{1-r}}\right), & \text{if } \alpha\geq r+1.\end{cases}.$$

*(vi)* $\mathbb{E}[\|X(t)-x^\star\|^2]=\mathcal{O}\left(\frac{1}{t^{1-r}}+\frac{1}{t^{\frac{r}{p}}}+t^rR(t)\right)$, *which goes to* $0$ *as* $t\to+\infty$ *if* $r\in]0,\frac{1}{2}]$. *Moreover, if* $\sigma_\infty^2(t)=\mathcal{O}(t^{-\alpha})$ *for* $\alpha>\max\{2r,1\}$, *then*

$$\mathbb{E}[\|X(t)-x^\star\|^2]=\mathcal{O}\left(\frac{1}{t^{1-r}}+\frac{1}{t^{\frac{r}{p}}}+\frac{1}{t^{\alpha-2r}}\right).$$

*In particular,*

$$\mathbb{E}[\|X(t)-x^\star\|^2]=\begin{cases}\mathcal{O}\left(\frac{1}{t^{1-r}}\right), & \text{if } r\in\left]\frac{p}{p+1},1\right[,\alpha>r+1;\\ \mathcal{O}\left(\frac{1}{t^{\frac{r}{p}}}\right), & \text{if } r\in\left]0,\frac{p}{p+1}\right[,\alpha>\max\{1,\frac{r(2p+1)}{p}\};\\ \mathcal{O}\left(\frac{1}{t^{\alpha-2r}}\right), & \text{if } r\in\left]\frac{p}{2p+1},1\right[,\alpha\in\left(\max\{2r,1\},\min\{r+1,\frac{r(2p+1)}{p}\}\right).\end{cases}$$

**Remark 4.12.** The expression in (ii) goes to $0$ as $t\to+\infty$ since $\lim_{t\to\infty}t\sigma_\infty^2(t)=0$ and $r<1$.

**Proof.** The existence and uniqueness of a solution was already stated in Theorem 4.1.

The first item is a direct consequence of Lemma A.4, for the second one we recall that $\sigma_\infty\in\mathrm{L}^2([t_0,+\infty[)$ and is non-increasing, and we proceed as follows:

$$\begin{aligned}R(t)&=e^{-\frac{t^{1-r}}{1-r}}\int_{t_1}^{\frac{t_1+t}{2}}e^{\frac{s^{1-r}}{1-r}}\sigma_\infty^2(s)ds+e^{-\frac{t^{1-r}}{1-r}}\int_{\frac{t_1+t}{2}}^{t}e^{\frac{s^{1-r}}{1-r}}\sigma_\infty^2(s)ds\\ &\leq e^{\left(\frac{t_0}{2}\right)^r}e^{-t^r(1-2^{-r})}\int_{t_1}^{+\infty}\sigma_\infty^2(s)ds+\sigma_\infty^2\left(\frac{t_1+t}{2}\right)D_{\frac{1}{1-r},1-r}(t),\end{aligned}$$

where

$$D_{a,b}(t)=e^{-at^b}\int_0^t e^{as^b}ds.$$

As a corollary of an upper bound of the Dawson integral shown in [27, Section 7.8], we have that

$$D_{a,b}(t)\leq\frac{2}{ab}t^{1-b},\quad 0<b\leq 2,a>0,t>0,$$

thus we obtain

$$R(t)=\mathcal{O}\left(\exp(-t^r(1-2^{-r}))+t^r\sigma_\infty^2\left(\frac{t_1+t}{2}\right)\right),$$

and $\lim_{t\to\infty} t^r \sigma_\infty^2(t) = 0$, to show this, we recall that $\sigma_\infty^2$ is non-increasing, then

$$0 \leq t\sigma_\infty^2(t) \leq 2\int_{\frac{t}{2}}^{t} \sigma_\infty^2(u)du,$$

and the right hand side goes to 0 as $t \to +\infty$ since $\sigma_\infty \in \mathrm{L}^2([t_0, +\infty[)$, thus we obtain that $\lim_{t\to\infty} t\sigma_\infty^2(t) = 0$ (as mentioned in Remark 4.12), and this directly implies the desired.

The rest of the proof follows by using Itô's formula with $\phi(t,x) = e^{\int_{t_1}^t \frac{ds}{s^r}} E(t,x)$, taking expectation and following the calculus done in [10, Theorem 5], we obtain the same results as in Theorem 4.9 (or [10, Theorem 5]) up to the term $R(t)$, therefore, this result could be seen as the stochastic counterpart of the mentioned Theorem. □

**Remark 4.13.** Tikhonov regularization implies strong convergence of the trajectory to the minimal norm solution, therefore, in the stochastic case you have to be careful in the tuning of the noise in order to not break this convergence. In the almost sure sense, you can tune appropriately the Tikhonov parameter without assuming more than $\sigma_\infty \in \mathrm{L}^2([t_0, +\infty[)$, nevertheless, in expectation, you may require a stronger assumption in general on the noise $\sigma_\infty^2$ in order to obtain a useful convergence rate, this is reflected in items (v) and (vi) of Theorem 4.11.

**Remark 4.14.** In the finite-dimensional case, i.e., $\mathbb{H} = \mathbb{R}^d$ (not necessarily $\mathbb{K}$), we can loosen the assumption $f \in C_L^2(\mathbb{H})$ to $f \in C_L^{1,1}(\mathbb{H})$ due to [1, Proposition 2.2].

# 5 Conclusion, Perspectives

The purpose of this work was to study the convergence properties of trajectories of subgradient-like flows under stochastic errors in infinite dimensions. The aim is to solve non-smooth convex optimization problems with noisy subgradient input with vanishing variance. In this regard, we have shown important properties of these trajectories, such as the almost sure behavior in the long run with and without Tikhonov regularization. This work let us study important extensions, among them, we mention the following ones:

- Extend Theorem 3.6 to the case of maximal monotone operators, i.e., replace $\nabla f$ by $A$ (a maximal monotone operator) and $\partial g$ by $B$ (a set-valued maximal monotone operator) in the dynamic and prove the almost sure weak convergence to a zero of $A + B$.
- Investigate the transition to second-order dynamics via time-scaling and averaging, and analyzing its corresponding convergence properties.
- Study second-order dynamics with inertia in view of understanding the behavior of accelerated dynamics in the presence of stochastic errors. This investigation will involve an independent Lyapunov analysis.

# A Auxiliary results

## A.1 Deterministic results

**Lemma A.1.** *Let $t_0 \geq 0$ and $a, b : [t_0, +\infty[\to \mathbb{R}_+$. If $\lim_{t\to\infty} a(t)$ exists, $b \notin \mathrm{L}^1([t_0, +\infty[)$ and $\int_{t_0}^{\infty} a(s)b(s)ds < +\infty$, then $\lim_{t\to\infty} a(t) = 0$.*

**Lemma A.2 (Comparison Lemma).** *Let $t_0 \geq 0$ and $T > t_0$. Assume that $h : [t_0, +\infty[ \to \mathbb{R}_+$ is measurable with $h \in \mathrm{L}^1([t_0, T])$, that $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and non-decreasing, $\varphi_0 > 0$ and the Cauchy problem*

$$\begin{cases} \varphi'(t) = -\psi(\varphi(t)) + h(t) & \textit{for almost all } t \in [t_0, T] \\ \varphi(t_0) = \varphi_0 \end{cases}$$

*has an absolutely continuous solution $\varphi : [t_0, T] \to \mathbb{R}_+$. If a bounded from below lower semicontinuous function $\omega : [t_0, T] \to \mathbb{R}_+$ satisfies*

$$\omega(t) \leq \omega(s) - \int_s^t \psi(\omega(\tau))d\tau + \int_s^t h(\tau)d\tau$$

*for $t_0 \leq s < t \leq T$ and $\omega(t_0) = \varphi_0$, then*

$$\omega(t) \leq \varphi(t) \quad \textit{for } t \in [t_0, T].$$

**Lemma A.3.** *Let $f : \mathbb{R}_+ \to \mathbb{R}$ and $\liminf_{t\to\infty} f(t) \neq \limsup_{t\to\infty} f(t)$. Then there exists a constant $\alpha$, satisfying $\liminf_{t\to\infty} f(t) < \alpha < \limsup_{t\to\infty} f(t)$, such that for every $\beta > 0$, we can define a sequence $(t_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ such that*

$$f(t_k) > \alpha, \quad t_{k+1} > t_k + \beta \quad \forall k \in \mathbb{N}.$$

**Proof.** See proof in [1, Lemma A.3]. ☐

**Lemma A.4.** *Take $t_0 > 0$, and let $f \in \mathrm{L}^1([t_0, +\infty[)$ be continuous. Consider a non-decreasing function $\varphi : [t_0, +\infty[ \to \mathbb{R}_+$ such that $\lim_{t\to+\infty} \varphi(t) = +\infty$. Then $\lim_{t\to+\infty} \frac{1}{\varphi(t)} \int_{t_0}^t \varphi(s)f(s)ds = 0$.*

**Proof.** See proof in [28, Lemma A.5] ☐

## A.2 Stochastic results

### A.2.1 On stochastic processes

Let us recall some elements of stochastic analysis. Throughout the paper, $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $\{\mathcal{F}_t | t \geq 0\}$ is a filtration of the $\sigma-$algebra $\mathcal{F}$. Given $\mathcal{C} \in \mathcal{P}(\Omega)$, we will denote $\sigma(\mathcal{C})$ the $\sigma-$algebra generated by $\mathcal{C}$. We denote $\mathcal{F}_\infty \stackrel{\text{def}}{=} \sigma\left(\bigcup_{t\geq 0} \mathcal{F}_t\right) \in \mathcal{F}$.

The expectation of a random variable $\xi : \Omega \to \mathbb{H}$ is denoted by

$$\mathbb{E}(\xi) \stackrel{\text{def}}{=} \int_\Omega \xi(\omega)d\mathbb{P}(\omega).$$

An event $E \in \mathcal{F}$ happens almost surely if $\mathbb{P}(E) = 1$, and it will be denoted as "$E$, $\mathbb{P}$-a.s." or simply "$E$, a.s.". The indicator function of an event $E \in \mathcal{F}$ is denoted by

$$\mathbb{1}_E(\omega) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } \omega \in E, \\ 0 & \text{otherwise.} \end{cases}$$

An $\mathbb{H}$-valued stochastic process starting at $t_0 \geq 0$ is a function $X : \Omega \times [t_0, +\infty[ \to \mathbb{H}$. It is said to be continuous if $X(\omega, \cdot) \in C([t_0, +\infty[; \mathbb{H})$ for almost all $\omega \in \Omega$. We will denote $X(t) \stackrel{\text{def}}{=} X(\cdot, t)$. We are going to study SDE's, and in order to ensure the uniqueness of a solution, we introduce a relation over stochastic processes. Two stochastic processes $X, Y : \Omega \times [t_0, T] \to \mathbb{H}$ are said to be equivalent if $X(t) = Y(t)$, $\forall t \in [t_0, T]$, $\mathbb{P}$-a.s. This leads us to define the equivalence relation $\mathcal{R}$, which associates the equivalent stochastic processes in the same class.

Furthermore, we will need some properties about the measurability of these processes. A stochastic process $X : \Omega \times [t_0, +\infty[ \to \mathbb{H}$ is progressively measurable if for every $t \geq t_0$, the map $\Omega \times [t_0, t] \to \mathbb{H}$ defined by $(\omega, s) \to X(\omega, s)$ is $\mathcal{F}_t \otimes \mathcal{B}([t_0, t])$-measurable, where $\otimes$ is the product $\sigma$-algebra and $\mathcal{B}$ is the Borel $\sigma$-algebra. On the other hand, $X$ is $\mathcal{F}_t$-adapted if $X(t)$ is $\mathcal{F}_t$-measurable for every $t \geq t_0$. It is a direct consequence of the definition that if $X$ is progressively measurable, then $X$ is $\mathcal{F}_t$-adapted.

Let us define the quotient space:

$$S^0_{\mathbb{H}}[t_0, T] \stackrel{\text{def}}{=} \{X : \Omega \times [t_0, T] \to \mathbb{H},\ X \text{ is a prog. measurable cont. stochastic process}\} \Big/ \mathcal{R}.$$

Set $S^0_{\mathbb{H}}[t_0] \stackrel{\text{def}}{=} \bigcap_{T \geq t_0} S^0_{\mathbb{H}}[t_0, T]$. For $\nu > 0$, we define $S^\nu_{\mathbb{H}}[t_0, T]$ as the subset of processes $X(t)$ in $S^0_{\mathbb{H}}[t_0, T]$ such that

$$S^\nu_{\mathbb{H}}[t_0, T] \stackrel{\text{def}}{=} \left\{ X \in S^0_{\mathbb{H}}[t_0, T] :\ \mathbb{E}\left( \sup_{t \in [t_0, T]} \|X_t\|^\nu \right) < +\infty \right\}.$$

We define $S^\nu_{\mathbb{H}}[t_0] \stackrel{\text{def}}{=} \bigcap_{T \geq t_0} S^\nu_{\mathbb{H}}[t_0, T]$.

Let $I \subseteq \mathbb{N}$ be a numerable set such that $\{e_i\}_{i \in I}$ is an orthonormal basis of $\mathbb{K}$, and $\{w_i(t)\}_{i \in I, t \geq 0}$ be a sequence of independent Brownian motions defined on the filtered space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. The process

$$W(t) = \sum_{i \in I} w_i(t) e_i$$

is well-defined (independent from the election of $\{e_i\}_{i \in I}$) and is called a $\mathbb{K}$-valued Brownian motion. Besides, let $G : \Omega \times \mathbb{R}_+ \to \mathcal{L}_2(\mathbb{K}; \mathbb{H})$ be a measurable and $\mathcal{F}_t$-adapted process, then we can define $\int_0^t G(s) dW(s)$ which is the stochastic integral of $G$, and we have that $G \to \int_0^\cdot G(s) dW(s)$ is an isometry between the measurable and $\mathcal{F}_t-$adapted $\mathcal{L}_2(\mathbb{K}; \mathbb{H})-$valued processes and the space of $\mathbb{H}$-valued continuous square-integrable martingales (see [23, Theorem 2.3]).

**Proposition A.5.** *(see [29] and [30, Section 1.2]) (Burkholder-Davis-Gundy Inequality) Let $p > 0$, $W$ be a $\mathbb{K}$-valued Brownian motion defined over a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ and $g : \Omega \times \mathbb{R}_+ \to \mathbb{K}$ a progressively measurable process (with our usual notation $g(t) \stackrel{\text{def}}{=} g(\cdot, t)$) such that*

$$\mathbb{E}\left[ \left( \int_0^T \|g(s)\|^2 ds \right)^{\frac{p}{2}} \right] < +\infty, \quad \forall T > 0.$$

*Then, there exists $C_p > 0$ (only depending on $p$) for every $T > 0$ such that:*

$$\mathbb{E}\left[ \sup_{t \in [0, T]} \left| \int_0^t \langle g(s), dW(s) \rangle \right|^p \right] \leq C_p \mathbb{E}\left[ \left( \int_0^T \|g(s)\|^2 ds \right)^{\frac{p}{2}} \right].$$

**Theorem A.6.** *Let $\mathbb{H}$ be a real separable Hilbert space and $(M_t)_{t \geq 0} : \Omega \to \mathbb{H}$ be a continuous martingale such that $\sup_{t \geq 0} \mathbb{E}\left( \|M_t\|^2 \right) < +\infty$. Then there exists a $\mathbb{H}-$valued random variable $M_\infty \in \mathrm{L}^2(\Omega; \mathbb{H})$ such that $\text{s-}\lim_{t \to \infty} M_t = M_\infty$ a.s..*

**Proof.** Consider $(M_k)_{k\in\mathbb{N}}$ to be the embedded discrete parameter martingale. Since $\sup_{k\in\mathbb{N}}\mathbb{E}\|M_k\|^2 < +\infty$, then $(M_k)_{k\in\mathbb{N}}$ is uniformly integrable and by [31, Theorem 3], there exists a measurable $\mathbb{H}$-valued random variable $M_\infty \in \mathrm{L}^2(\Omega;\mathbb{H})$ such that $\lim_{k\to\infty}\|M_k - M_\infty\| = 0$ a.s.. In turn, using the dominated convergence theorem (see [32, Theorem 1.34]), we also have

$$\lim_{k\to\infty}\mathbb{E}(\|M_k - M_\infty\|^2) = 0. \tag{A.1}$$

The rest of the proof is inspired by the arguments in the proof of [33, Theorem 2.2].

We consider an arbitrary $k \in \mathbb{N}^*$ and $\delta > 0$. Since $(M_{t+k} - M_k)_{t\geq 0}$ is also a $\mathbb{H}-$valued martingale, we can use Doob's maximal inequalities for $\mathbb{H}-$valued martingales shown in [23, Theorem 2.2], which gives us

$$\delta^2\mathbb{P}\left(\sup_{s\in[0,t]}\|M_{s+k} - M_k\| > \delta\right) \leq \mathbb{E}(\|M_{t+k} - M_k\|^2). \tag{A.2}$$

Let $n \in \mathbb{N}^*$ be arbitrary. We have

$$\mathbb{P}\left(\sup_{s\in\mathbb{Q}\cap[0,n]}\|M_{s+k} - M_\infty\| > \delta\right) \leq \mathbb{P}\left(\sup_{s\in\mathbb{Q}\cap[0,n]}\|M_{s+k} - M_k\| > \frac{\delta}{2}\right) + \mathbb{P}\left(\|M_k - M_\infty\| > \frac{\delta}{2}\right).$$

Using (A.2) and Markov's inequality, we get the bound

$$\begin{aligned}\delta^2\mathbb{P}\left(\sup_{s\in\mathbb{Q}\cap[0,n]}\|M_{s+k} - M_\infty\| > \delta\right) &\leq 4\mathbb{E}(\|M_{n+k} - M_k\|^2) + 4\mathbb{E}(\|M_k - M_\infty\|^2)\\ &\leq 8\mathbb{E}(\|M_{n+k} - M_\infty\|^2) + 12\mathbb{E}(\|M_k - M_\infty\|^2).\end{aligned} \tag{A.3}$$

In turn, we get

$$\begin{aligned}\delta^2\mathbb{P}\left(\sup_{s\in\mathbb{Q},s\geq k}\|M_s - M_\infty\| > \delta\right) &\leq \delta^2\mathbb{P}\left(\bigcup_{n\in\mathbb{N}^*}\left\{\sup_{s\in\mathbb{Q}\cap[0,n]}\|M_{s+k} - M_\infty\| > \delta\right\}\right)\\ &\leq \delta^2\liminf_{n\to\infty}\mathbb{P}\left(\sup_{s\in\mathbb{Q}\cap[0,n]}\|M_{s+k} - M_\infty\| > \delta\right)\\ &\leq 12\mathbb{E}\left(\|M_k - M_\infty\|^2\right),\end{aligned}$$

where we have used (A.3) and in the last inequality, that $\lim_{n\to\infty}\mathbb{E}(\|M_{n+k} - M_\infty\|^2) = 0$ by (A.1). Taking $k\to\infty$, and using again (A.1), we conclude that for all $\delta > 0$

$$\lim_{k\to\infty}\mathbb{P}\left(\sup_{s\in\mathbb{Q},s\geq k}\|M_s - M_\infty\| > \delta\right) = 0.$$

For $k \in \mathbb{N}^*$, we define $A_k \stackrel{\text{def}}{=} \{\omega \in \Omega : \sup_{s\in\mathbb{Q},s\geq k}\|M_s(\omega) - M_\infty(\omega)\| > \delta\}$, since $(A_k)_{k\in\mathbb{N}^*}$ is a non-increasing sequence of sets:

$$0 = \lim_{k\to\infty}\mathbb{P}(A_k) = \mathbb{P}\left(\bigcap_{k\in\mathbb{N}^*}A_k\right).$$

Defining for $l \geq 0$, $D_l = \{\omega \in \Omega : \|M_l(\omega) - M_\infty(\omega)\| > \delta\}$, it is direct to check that $\bigcup_{l \geq k, l \in \mathbb{Q}} D_l \subseteq A_k$ for every $k \in \mathbb{N}^*$. Therefore, we obtain that

$$\mathbb{P}\left( \bigcap_{k \in \mathbb{N}^*} \bigcup_{l \geq k, l \in \mathbb{Q}} D_l \right) = 0,$$

which is equivalent to $\text{s-}\lim_{s \to \infty, s \in \mathbb{Q}} M_t = M_\infty$ a.s.. The result follows from classical arguments of continuity of the martingale. □

**Theorem A.7.** *[24, Theorem 1.3.9] Let $\{A_t\}_{t \geq 0}$ and $\{U_t\}_{t \geq 0}$ be two continuous adapted increasing processes with $A_0 = U_0 = 0$ a.s.. Let $\{M_t\}_{t \geq 0}$ be a real-valued continuous local martingale with $M_0 = 0$ a.s.. Let $\xi$ be a nonnegative $\mathcal{F}_0$-measurable random variable. Define*

$$X_t = \xi + A_t - U_t + M_t \quad \text{for } t \geq 0.$$

*If $X_t$ is nonnegative and $\lim_{t \to \infty} A_t < \infty$, then $\lim_{t \to \infty} X_t$ exists and is finite, and $\lim_{t \to \infty} U_t < \infty$.*

# References


[1] Rodrigo Maulen-Soto, Jalal Fadili, and Hedy Attouch. An SDE perspective on stochastic convex optimization. *arXiv:2207.02750*, 2022.

[2] R. Cominetti, J. Peypouquet, and S. Sorin. Strong asymptotic convergence of evolution equations governed by maximal monotone operators with tikhonov regularization. *J. Differential Equations*, 245:3753–3763, 2008.

[3] A. Cauchy. Méthode générale pour la résolution des systèmes d'équations simultanées. *Comptes Rendus de l'Académie des Sciences de Paris*, 25:536–538, 1847.

[4] N.Z. Shor. Minimization methods for non-differentiable functions. *Springer Series in Computational Mathematics*, 1985.

[5] M. Akgul. Topics in relaxation and ellipsoidal methods. *Research Notes in mathematics*, 97, 1984.

[6] Claude Lemaréchal. Lagrangian relaxation. *Computational combinatorial optimization: Papers from the Spring School held in Schloß Dagstuhl*, 2241:112–156, 2001.

[7] Kiwiel and C Krzysztof. Lagrangian relaxation via ballstep subgradient methods. *Mathematics of Operations Research*, 32 (3):669–686, 2007.

[8] Herbert Robins and Sutton Monro. A stochastic approximation method. *Ann. Math. Statist. 22*, pages 400–407, 1951.

[9] J. Gwinner. Bibliography on nondifferentiable optimization and non-smooth analysis. *J. Comput. Appl. Math*, 7:277–285, 1981.

[10] H. Attouch, Z. Chbani, and H. Riahi. Accelerated gradient methods with strong convergence to the minimum norm minimizer: a dynamic approach combining time scaling, averaging, and tikhonov regularization. *arXiv:2211.10140v1 [math.OC]*, 2022.

[11] R.T. Rockafellar. Convex analysis. *Princeton univerisity press*, 28, 1997.

[12] F. E. Browder. Existence and approximation of solutions of nonlinear variational inequalities. *Proc. Nat. Acad. Sci. U.S.A*, 56:1080–1086, 1966.

[13] F. E. Browder. Convergence of approximants to fixed points of nonexpansive nonlinear mappings in Banach spaces. *Arch. Rational Mech. Anal.*, 24:82–90, 1967.

[14] H. Attouch and M.-O Czarnecki. Asymptotic behavior of coupled dynamical systems with multiscale aspects. *J. Differential Equations*, 248.6:1315–1344, 2010.

[15] Michal Kisielewicz. Sstochastic differential inclusions and applications. *Springer Optimization and its applications*, 80, 2013.

[16] Roger Pettersson. Yosida approximations for multivalued stochastic differential equations. *Stochastics and Stochastics reports*, 52:107–120, 1994.

[17] G. Bocsan. On Wiener stochastic integrals of multifunctions. *Univ. Tim. FSN*, 87:1–7, 1987.

[18] J.P Aubin and G. Da Prato. The viability theorem for stochastic differential inclusions. *Stoch. Anal. Appl.*, 16:1–15, 1998.

[19] Aurel Rascanu. Deterministic and stochastic differential equations in Hilbert spaces involving multivalued maximal monotone operators. *arXiv:1402.0748*, 2014.

[20] T.E. Govindan. Yosida approximations of stochastic differential equations in infinite dimensions and applications. *Springer: Probability Theory and Stochastic modelling*, 79, 2016.

[21] Haim Brézis. Opérateurs maximaux monotones et semi-groupes de contractions dans les espace Hilbert. *North-Holland mathematics studies; 5.*, 1973.

[22] Desmond J. Higham, Xuerong Mao, and Andrew M. Stuart. Strong convergence of Euler-type methods for nonlinear stochastic differential equations. *SIAM J. Numer. Anal., 40(3)*, page 1041–1063, 2006.

[23] Leszek Gawarecki and Vidyadhar Mandrekar. Stochastic differential equations in infinite dimensions. *Springer*, 2011.

[24] Xuerong Mao. Stochastic differential equations and applications. *Elsevier*, 2007.

[25] Z. Opial. Weak convergence of the sequence of successive approximations for nonexpansive mappings. *Bull. Amer. Math. Soc.*, 73:591–597, 1967.

[26] D. Torralba. Convergence épigraphique et changements d'échelle en analyse variationelle et optimisation. *PhD Thesis, Université de Montpellier 2*, page 160, 1996.

[27] F.W.J. Olver, D.w. Lozier, R.F Boisvert, and Clarck C.W. Nist handbook of mathematical functions. *Cambridge University Press*, 2010.

[28] Hedy Attouch, Jalal Fadili, and Vyacheslav Kungurtsev. On the effect of perturbations in first-order optimization methods with inertia and hessian driven damping. *Evolution equations and Control*, 12(1):71–117, 2023.

[29] D.L. Burkholder, B. Davis, and R.F. Gundy. Integral inequalities for convex functions of operators on martingales. *Proc. 6th Berkeley Symp. Math. Statistics and Probability*, 2:223–240, 1972.

[30] Aurel Rascanu and Eduard Rotenstein. The Fitzpatrick function-a bridge betweem convex analysis and multivalued stochastic differential equations. *arXiv:0809.4447*, 2009.

[31] Frank S. Scalora. Abstract martingale convergence theorems. *Pacific J. Math.*, 11 (1):347–374, 1961.

[32] Walter Rudin. Real and complex analysis. *McGraw-Hill*, 1987.

[33] Joe Ghafari. A proof and an application of the continuous parameter martingale convergence theorem. *Research in Statistics*, 1 (1), 2023.