Inertial Methods with Viscous and Hessian driven Damping for Non-Convex Optimization

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July 23, 2024*

Abstract. In this paper, we aim to study non-convex minimization problems via second-order (in-time) dynamics, including a non-vanishing viscous damping and a geometric Hessian-driven damping. Second-order systems that only rely on a viscous damping may suffer from oscillation problems towards the minima, while the inclusion of a Hessian-driven damping term is known to reduce this effect without explicit construction of the Hessian in practice. There are essentially two ways to introduce the Hessian-driven damping term: explicitly or implicitly. For each setting, we provide conditions on the damping coefficients to ensure convergence of the gradient towards zero. Moreover, if the objective function is definable, we show global convergence of the trajectory towards a critical point as well as convergence rates. Besides, in the autonomous case, if the objective function is Morse, we conclude that the trajectory converges to a local minimum of the objective for almost all initializations. We also study algorithmic schemes for both dynamics and prove all the previous properties in the discrete setting under proper choice of the stepsize.

Key words. Optimization; Inertial gradient systems, Time-dependent viscosity, Hessian Damping, Convergence of the trajectory, Łojasiewicz inequality, KŁ inequality, Convergence rate, Asymptotic behavior.

AMS subject classifications. 37N40, 46N10, 49M15, 65B99, 65K05, 65K10, 90B50, 90C26, 90C53

1 Introduction

1.1 Problem statement

Let us consider the minimization problem

$$\min_{x\in\mathbb{R}^d} f(x), \tag{P}$$

where the objective function $f:\mathbb{R}^d\to\mathbb{R}$ satisfies the following standing assumptions:

$$\begin{cases} f\in C^2(\mathbb{R}^d); \\ \inf f > -\infty. \end{cases} \tag{H_0}$$

Since the objective function is potentially non-convex, the problem (P) is NP-Hard. However, there are tractable methods to ensure theoretical convergence to a critical point, or even to a local minimizer. In this regard, a fundamental dynamic to consider is the gradient flow system:

*The insight and motivation for the study of inertial methods with viscous and Hessian driven damping came from the inspiring collaboration with our beloved friend and colleague Hedy Attouch before his unfortunate recent departure. We hope this paper is a valuable step in honoring his legacy.

$$\begin{cases} \dot{x}(t) + \nabla f(x(t)) = 0, \quad t > 0; \\ x(0) = x_0. \end{cases} \tag{GF}$$

For any bounded solution of (GF), using LaSalle's invariance principle, we check that $\lim_{t\to+\infty} \nabla f(x(t)) = 0$. If $d = 1$, any bounded solution of (GF) tends to a critical point. For $d \geq 2$ this becomes false in general, as shown in the counterexample by [1]. In order to avoid such behaviors, it is necessary to work with functions that present a certain structure. An assumption that will be central in our paper for the study of our dynamics and algorithms is that the function f satisfies the Kurdyka-Łojasiewicz (KŁ) inequality [2, 3, 4], which means, roughly speaking, that f is sharp up to a reparametrization. The KŁ inequality, including in its nonsmooth version, has been successfully used to analyze the asymptotic behavior of various types of dynamical systems [5, 6, 7, 8] and algorithms [9, 10, 11, 12, 13, 14, 15, 16, 17][1]. The importance of the KŁ inequality comes from the fact that many problems encountered in optimization involve functions satisfying such an inequality, and it is often elementary to check that the latter is satisfied; *e.g.* real semialgebraic/analytic functions [2, 3], functions definable in an o-minimal structure and more generally tame functions [4, 18].

The dynamic (GF) is known to yield a convergence rate of $\mathcal{O}(t^{-1})$ of the values in the convex setting (in fact even $o(t^{-1})$). Second-order inertial dynamical systems have been introduced to provably accelerate the convergence behavior in the convex case. They typically take the form

$$\ddot{x}(t) + \gamma(t)\dot{x}(t) + \nabla f(x(t)) = 0, \quad t > t_0, \tag{IGS$_\gamma$}$$

where $t_0 > 0$, $\gamma : [t_0, +\infty[\to \mathbb{R}_+$ is a time-dependent viscosity coefficient. An abundant literature has been devoted to the study of the inertial dynamics (IGS$_\gamma$). The importance of working with a time-dependent viscosity coefficient to obtain acceleration was stressed by several authors; see *e.g.* [19]. In particular, the case $\gamma(t) = \frac{\alpha}{t}$ was considered by Su, Boyd, and Candès [20], who were the first to show the rate of convergence $\mathcal{O}(t^{-2})$ of the values in the convex setting for $\alpha \geq 3$, thus making the link with the accelerated gradient method of Nesterov [21]. For $\alpha > 3$, an even better rate of convergence with little-o instead of big-$\mathcal{O}$ can be obtained together with global convergence of the trajectory; see [22, 23] and [24] for the discrete algorithmic case.

Another remarkable instance of (IGS$_\gamma$) corresponds to the well-known Heavy Ball with Friction (HBF) method, where $\gamma(t)$ is a constant, first introduced (in its discrete and continuous form) by Polyak in [25]. When f is strongly convex, it was shown that the trajectory converges exponentially with an optimal convergence rate if γ is properly chosen as a function of the strong convexity modulus. The convex case was later studied in [26] with a convergence rate on the values of only $\mathcal{O}(t^{-1})$. When f is non-convex, HBF was investigated both for the continuous dynamics [27, 28, 29, 30] and discrete algorithms [31, 17, 32].

However, because of the inertial aspects, (IGS$_\gamma$) may exhibit many small oscillations which are not desirable from an optimization point of view. To remedy this, a powerful tool consists in introducing into the dynamic a geometric damping driven by the Hessian of f. This gives the Inertial System with Explicit Hessian Damping which reads

$$\begin{cases} \ddot{x}(t) + \gamma(t)\dot{x}(t) + \beta(t)\nabla^2 f(x(t))\dot{x}(t) + \nabla f(x(t)) = 0, \quad t > t_0; \\ x(t_0) = x_0, \quad \dot{x}(t_0) = v_0, \end{cases} \tag{ISEHD}$$

[1]This list is by no means exhaustive.

where $\gamma, \beta : [t_0, +\infty[\to \mathbb{R}_+$. (ISEHD) was proposed in [33] (see also [34]). The second system we consider, inspired by [35] (see also [36] for a related autonomous system) is

$$\begin{cases} \ddot{x}(t) + \gamma(t)\dot{x}(t) + \nabla f(x(t) + \beta(t)\dot{x}(t)) = 0, \quad t > t_0; \\ x(t_0) = x_0, \quad \dot{x}(t_0) = v_0. \end{cases} \tag{ISIHD}$$

(ISIHD) stands for Inertial System with Implicit Hessian Damping. The rationale behind the use of the term "implicit" comes from a Taylor expansion of the gradient term (as $t \to +\infty$ we expect $\dot{x}(t) \to 0$) around $x(t)$, which makes the Hessian damping appear indirectly in (ISIHD). Following the physical interpretation of these two ODEs, we call the non-negative parameters γ and β the viscous and geometric damping coefficients, respectively. The two ODEs (ISEHD) and (ISIHD) were found to have a smoothing effect on the energy error and oscillations [35, 37, 38]. Moreover, in [39] they obtain fast convergence rates for the values for the two ODEs when $\gamma(t) = \frac{\alpha}{t}$ ($\alpha > 3$) and $\beta(t) = \beta > 0$. However, the previous results are exclusive to the convex case. Nevertheless, in [40], the authors analyzes (ISEHD) with constant viscous and geometric damping in a non-convex setting, concluding that the bounded solution trajectories converges to a critical point of the objective under Łojasiewicz inequality and giving sublinear convergence rates. Moreover, in [41] the author shows that the previous dynamic avoids strict saddle points when the objective is Morse. In these two works they propose a discretization called INNA, where the conditions on the stepsize are more stringent than ours but let them consider arbitrary values for the geometric damping. To the best of our knowledge, there is no further analysis of (ISEHD) and (ISIHD) when the objective is non-convex and definable. In this work, our goal is to fill this gap.

1.2 Contributions

In this paper, we analyze the convergence properties of (ISEHD) and (ISIHD) when f is non-convex, and when the time-dependent viscosity coefficient γ is non-vanishing and the geometric damping is constant, *i.e.* $\beta(t) \equiv \beta \geq 0$. We will also propose appropriate discretizations of these dynamics and establish the corresponding convergence guarantees for the resulting discrete algorithms. More precisely, our main contributions can be summarized as follows:

- We provide a Lyapunov analysis of (ISEHD) and (ISIHD) and show convergence of the gradient to zero and convergence of the values. Moreover, assuming that the objective function is definable, we prove the convergence of the trajectory to a critical point (see Theorem 3.1 and 4.1). Furthermore, when f is also Morse, using the center stable manifold theorem, we establish a generic convergence of the trajectory to a local minimum of f (see Theorem 3.3 and 4.2).
- We provide convergence rates of (ISEHD) and (ISIHD) and show that they depend on the desingularizing function of the Lyapunov energy (see Theorems 3.4 and 4.3).
- By appropriately discretizing (ISEHD) and (ISIHD), we propose algorithmic schemes and prove the corresponding convergence properties, which are the counterparts of the ones shown for the continuous-time dynamics. In particular, assuming that the objective function is definable, we show global convergence of the iterates of both schemes to a critical point of f. Furthermore, a generic strict saddle points (see Definition 2.3) avoidance result will also be proved (see Theorem 3.7 and 4.5).
- Convergence rates for the discrete schemes will also be established under the Łojasiewicz property where the convergence rate will be shown to depend on the exponent of the desingularizing function (see Theorem 3.9 and 4.6).

We will report a few numerical experiments to support the above findings.

1.3 Relation to prior work

Continuous-time dynamics. There is abundant literature regarding the dynamics (ISEHD) and (ISIHD), either in the exact case or with deterministic errors, but solely in the convex case; see [35, 39, 27, 42, 43, 44, 45, 46, 47, 48, 49, 50]). Nevertheless, to the best of our knowledge, the non-convex setting was still open until now.

Heavy ball-like methods. As mentioned before, the HBF method, was first introduced by Polyak in [25] where linear convergence was shown for f strongly convex. Convergence rates of HBF taking into account the geometry of the objective can be found in [12] for the convex case and [32] for the non-convex case.

In [51], the authors study the system

$$\ddot{x}(t) + G(\dot{x}(t)) + \nabla f(x(t)) = 0,$$

where $G : \mathbb{R}^d \to \mathbb{R}^d$ is such that $\langle G(v), v\rangle \geq c\|v\|^2$ and $\|G(v)\| \leq C\|v\|, \forall v \in \mathbb{R}^d$, for some $0 < c \leq C$. They show that when f is real analytic (hence verifies the Łojasiewicz inequality), the trajectory converges to a critical point. To put it in our terms, the analysis is equivalent to study (IGS_γ) in the case where there exists $c, C > 0$ such that $0 < c \leq \gamma(t) \leq C$ for every $t \geq 0$ (see assumption (H_γ)), letting us to conclude as in [51]. We will extend this result to the systems (ISEHD) and (ISIHD) which will necessitate new arguments.

HBF was also studied in the non-convex case by [8] where it was shown that it can be viewed as a quasi-gradient system. They also proved that a desingularizing function of the objective desingularizes the total energy and its deformed versions. They used this to establish the convergence of the trajectory for instance in the definable case. Global convergence of the HBF trajectory was also shown in [28] when f is a Morse function [2].

Inertial algorithms. In the literature regarding algorithmic schemes that include inertia in the non-convex setting, we can mention: [53] which proposes a multi-step inertial algorithm using Forward-Backward for non-convex optimization. In [29] they study quasiconvex functions and use an implicit discretization of HBF to derive a proximal algorithm. In [17], they introduce an inertial Forward-Backward splitting method, called iPiano-a generalization of HBF, they show convergence of the values and a convergence rate of the algorithm. Besides, in [32], the author presents local convergence results for iPiano. Then, in [54, 55] several abstract convergence theorems are presented to apply them to different versions of iPiano. In [56, 57] they propose inertial algorithms of different nature (Tseng's type and Forward-Backward, respectively) for non-convex and non-smooth optimization. In the previous works, they assume KŁ-like inequality to conclude with some type of convergence (weak or strong) towards a critical point of the objective.

In this sense, the closest work to ours is [40], where they study (ISEHD) when the viscous and geometric dampings are positive constants, and a discretization of this dynamic called INNA. Assuming that the objective function is semi-algebraic and the solution trajectories (resp. iterations) are bounded, their main results are that: both the dynamic and the discretization converge to critical points of the objective (using approximation theory), and that the continuous dynamic exhibits a sublinear convergence rate. Our work extends beyond this, it considers not only the explicit version (ISEHD) but also the implicit one (ISIHD). Even though the choice of the value of the geometric damping is less general than the one presented in [40], we allow for variable viscous damping, propose a less stringent discretization regarding the stepsize, and conclude with the convergence of the dynamic to a critical point under general KŁ property on the objective,

[2] It turns out that a Morse function satisfies the Łojasiewicz inequality with exponent $1/2$; see [52].

which includes semi-algebraic functions as a special case. Additionally, we show convergence rates under KŁ inequality. Moreover, we propose new algorithms and an independent analysis for the discrete setting, showing convergence of the proposed algorithms to critical points under KŁ-inequality, and we also exhibit convergence rates of the proposed algorithms under Łojasiewicz inequality.

Trap avoidance. In the non-convex setting, generic strict saddle point avoidance of descent-like algorithms has been studied by several authors building on the (center) stable manifold theorem [58, Theorem III.7] which finds its roots in the work of Poincaré. Genericity is in general either with respect to initialization or with respect to random perturbation. Note that genericity results for quite general systems, even in infinite dimensional spaces, is an important topic in the dynamical system theory; see *e.g.* [59] and references therein.

First-order descent methods can circumvent strict saddle points provided that they are augmented with unbiased noise whose variance is sufficiently large in each direction. Here, the seminal works of [60] and [61] allow to establish that the stochastic gradient descent (and more generally the Robbins-Monro stochastic approximation algorithm) avoids strict saddle points almost surely. Those results were extended to the discrete version of HBF by [62] who showed that perturbation allows to escape strict saddle points, and it does so faster than gradient descent. In [63] and [64], the authors analyze a stochastic version of HBF (in continuous-time), showing convergence towards a local minimum under different conditions on the noise. In this paper, we only study genericity of trap avoidance with respect to initialization.

Recently, there has been active research on how gradient-type descent algorithms escape strict saddle points generically on initialization; see *e.g.* [65, 66, 67, 68] and references therein. In [29], the authors were concerned with HBF and showed that if the objective function is C^2, coercive and Morse, then generically on initialization, the solution trajectory converges to a local minimum of f. A similar result is also stated in [28]. The algorithmic counterpart of this result was established in [69] who proved that the discrete version of HBF escapes strict saddles points for almost all initializations.

We have to mention that the closest work in this regard is [41], where the author studies (ISEHD) when the viscous and geometric dampings are positive constants and the discretization INNA proposed in [40]. Using the continuous and discrete version of the global stable manifold theorem, the author shows that both the continuous-time dynamic (when the objective is Morse) and the INNA algorithm almost always avoids strict saddle points.

Our goal in this paper is to establish the same kind of results for (ISEHD) and (ISIHD) with variable, non-vanishing viscous damping and constant geometric damping, as well as for the proposed algorithms ((ISEHD-Disc), (ISIHD-Disc)). We would like to point out that the proof for the continuous-time case will necessitate (as in [41]) a more stringent assumption on the class of functions, for instance, that f is Morse, while this is not necessary for the discrete algorithms.

1.4 Organization of the paper

Section 2 introduces notations and recalls some preliminaries that are essential to our exposition. Sections 3 and 4 include the main contributions of our paper, establishing convergence of the trajectory and of the iterates under KŁ inequality, convergence rates and trap avoidance results. Section 5 is devoted to the numerical experiments.

2 Notation and preliminaries

We denote by $\mathbb{R}_+$ the set $[0, +\infty[$. Moreover, we denote $\mathbb{R}_+^*$ and $\mathbb{N}^*$ to refer to $\mathbb{R}_+ \setminus \{0\}$ and $\mathbb{N} \setminus \{0\}$, respectively. The finite-dimensional space $\mathbb{R}^d$ ($d \in \mathbb{N}^*$) is endowed with the canonical scalar product $\langle \cdot, \cdot \rangle$ whose norm is denoted by $\|\cdot\|$. We denote $\mathbb{R}^{n\times d}$ ($n, d \in \mathbb{N}^*$) the space of real matrices of dimension $n \times d$. We denote by $C^n(\mathbb{R}^d)$ the class of n-times continuously differentiable functions on $\mathbb{R}^d$. For a function $g : \mathbb{R}^d \to \mathbb{R}$ and $a, b \in \mathbb{R}$ such that $a < b$, we denote $[a < f < b]$ to the sublevel set $\{x \in \mathbb{R}^d : a < f(x) < b\}$. For a differentiable function $g : \mathbb{R}^d \to \mathbb{R}$, we will denote its gradient as ∇g, the set of its critical points as:

$$\mathrm{crit}(g) \stackrel{\text{def}}{=} \{u : \nabla g(u) = 0\},$$

and when g is twice differentiable, we will denote its Hessian as $\nabla^2 g$. For a differentiable function $G : \mathbb{R}^d \to \mathbb{R}^n$ we will denote its Jacobian matrix as $J_G \in \mathbb{R}^{n\times d}$. Let $A \in \mathbb{R}^{d\times d}$, then we denote $\lambda_i(A) \in \mathbb{C}$ the $i-$th eigenvalue of A, when A is symmetric then the eigenvalues are real and we denote $\lambda_{\min}(A), \lambda_{\max}(A)$ to be the minimum and maximum eigenvalue of A, respectively. If every eigenvalue of A is positive (resp. negative), we will say that A is positive (resp. negative) definite. We denote by I_d the identity matrix of dimensions $d \times d$ and $0_{n\times d}$ the null matrix of dimensions $n \times d$, respectively. The following lemma is essential to some arguments presented in this work:

Lemma 2.1. *[70, Theorem 3] Consider $A, B, C, D \in \mathbb{R}^{d\times d}$ and such that $AC = CA$. Then*

$$\det\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right) = \det(AD - CB).$$

And the following definitions will be important throughout the paper:

Definition 2.2 (Local extrema and saddle points). Consider a function $f \in C^2(\mathbb{R}^d)$. We will say that $\hat{x}$ is a local minimum (resp. maximum) of f if $\hat{x} \in \mathrm{crit}(f)$, $\nabla^2 f(\hat{x})$ is positive (resp. negative) definite. If $\hat{x}$ is a critical point that is neither a local minimum nor a local maximum, we will say that $\hat{x}$ is a saddle point of f.

Definition 2.3 (Strict saddle point). Consider a function $f \in C^2(\mathbb{R}^d)$, we will say that $\hat{x}$ is a strict saddle point of f if $\hat{x} \in \mathrm{crit}(f)$ and $\lambda_{\min}(\nabla^2 f(\hat{x})) < 0$.

Remark 2.4. According to this definition, local maximum points are strict saddles.

Definition 2.5 (Strict saddle property). A function $f \in C^2(\mathbb{R}^d)$ will satisfy the strict saddle property if every critical point is either a local minimum or a strict saddle.

This property is a reasonable assumption for smooth minimization. In practice, it holds for specific problems of interest, such as low-rank matrix recovery and phase retrieval. Moreover, as a consequence of Sard's theorem, for a full measure set of linear perturbations of a function f, the linearly perturbed function satisfies the strict saddle property. Consequently, in this sense, the strict saddle property holds generically in smooth optimization. We also refer to the discussion in [66, Conclusion].

Definition 2.6 (Morse function). A function $f \in C^2(\mathbb{R}^d)$ will be Morse if it satisfies the following conditions:

(i) For each critical point $\hat{x}$, $\nabla^2 f(\hat{x})$ is nonsingular.

(ii) There exists a nonempty set $I \subseteq \mathbb{N}$ and $(\hat{x}_k)_{k\in I}$ such that $\mathrm{crit}(f) = \bigcup_{k\in I}\{\hat{x}_k\}$.

Remark 2.7. By definition, a Morse function satisfies the strict saddle property.

Remark 2.8. Morse functions can be shown to be generic in the Baire sense in the space of C^2 functions; see [71].

Definition 2.9 (Desingularizing function). For $\eta > 0$, we consider

$$\kappa(0,\eta) \stackrel{\text{def}}{=} \{\psi : C^0([0,\eta[) \cap C^1(]0,\eta[) \to \mathbb{R}_+, \psi' > 0 \text{ on }]0,\eta[, \psi(0) = 0, \text{ and } \psi \text{ concave}\}.$$

Remark 2.10. The concavity property of the functions in $\kappa(0,\eta)$ is only required in the discrete setting.

Definition 2.11. If $f : \mathbb{R}^d \to \mathbb{R}$ is differentiable and satisfies the KŁ inequality at $\bar{x} \in \mathbb{R}^d$, then there exists $r, \eta > 0$ and $\psi \in \kappa(0,\eta)$, such that

$$\psi'(f(x) - f(\bar{x}))\|\nabla f(x) - \nabla f(\bar{x})\| \geq 1, \quad \forall x \in B(\bar{x}, r) \cap [f(\bar{x}) < f < f(\bar{x}) + \eta]. \tag{2.1}$$

Definition 2.12. A function $f : \mathbb{R}^d \to \mathbb{R}$ will satisfy the Łojasiewicz inequality with exponent $q \in]0,1]$ if f satisfies the KŁ inequality with $\psi(s) = c_0 s^{1-q}$ for some $c_0 > 0$.

It remains now to identify a broad class of functions f that verifies the KŁ inequality. A rich family is provided by semi-algebraic functions, *i.e.*, functions whose graph is defined by some Boolean combination of real polynomial equations and inequalities [72]. Such functions satisfy the Łojasiewicz property with $q \in [0,1[\cap \mathbb{Q}$; see [2, 3]. An even more general family is that of definable functions on an o-minimal structure over $\mathbb{R}$, which corresponds in some sense to an axiomatization of some of the prominent geometrical and stability properties of semi-algebraic geometry [73, 74]. An important result by Kurdyka in [4] showed that definable functions satisfy the KŁ inequality at every $\bar{x} \in \mathbb{R}^d$.

Remark 2.13. Morse functions verify the Łojasiewicz inequality with exponent $q = 1/2$; see [52].

3 Inertial System with Explicit Hessian Damping

Throughout the paper we will consider (ISEHD) and (ISIHD) with $t_0 = 0$. Also we will assume that the viscous damping $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is such that $\exists c, C > 0$, $c \leq C$, and

$$c \leq \gamma(t) \leq C, \quad \forall t \geq 0. \tag{H$_\gamma$}$$

Moreover, throughout this work, we consider a constant geometric damping, *i.e.* $\beta(t) \equiv \beta > 0$.

3.1 Continuous-time dynamics

Let us consider (ISEHD), as in [39], we will say that $x : \mathbb{R}_+ \to \mathbb{R}^d$ is a solution trajectory of (ISEHD) with initial conditions $x(0) = x_0, \dot{x}(0) = v_0$, if and only if, $x \in C^2(\mathbb{R}_+; \mathbb{R}^d)$ and there exists $y \in C^1(\mathbb{R}_+; \mathbb{R}^d)$ such that (x, y) satisfies:

$$\begin{cases} \dot{x}(t) + \beta\nabla f(x(t)) - \left(\frac{1}{\beta} - \gamma(t)\right)x(t) + \frac{1}{\beta}y(t) & = 0, \\ \dot{y}(t) - \left(\frac{1}{\beta} - \gamma(t) - \beta\gamma'(t)\right)x(t) + \frac{1}{\beta}y(t) & = 0, \end{cases} \tag{3.1}$$

with initial conditions $x(0) = x_0, y(0) = y_0 \stackrel{\text{def}}{=} -\beta(v_0 + \beta\nabla f(x_0)) + (1 - \beta\gamma(0))x_0$.

3.1.1 Global convergence of the trajectory

Our first main result is the following theorem.

Theorem 3.1. *Assume that* $0 < \beta < \frac{2c}{C^2}$, $f : \mathbb{R}^d \to \mathbb{R}$ *satisfies* (H_0), *and* $\gamma \in C^1(\mathbb{R}_+; \mathbb{R}_+)$ *obeys* (H_γ). *Consider* (ISEHD) *in this setting, then the following holds:*

(i) There exists a global solution trajectory $x : \mathbb{R}_+ \to \mathbb{R}^d$ *of* (ISEHD).

(ii) We have that $\nabla f \circ x \in L^2(\mathbb{R}_+; \mathbb{R}^d)$, *and* $\dot{x} \in L^2(\mathbb{R}_+; \mathbb{R}^d)$.

(iii) If we suppose that the solution trajectory x *is bounded over* $\mathbb{R}_+$, *then*

$$\lim_{t\to+\infty} \|\nabla f(x(t))\| = \lim_{t\to+\infty} \|\dot{x}(t)\| = 0,$$

and $\lim_{t\to+\infty} f(x(t))$ *exists.*

(iv) In addition to (iii), if we also assume that f *is definable, then* $\dot{x} \in L^1(\mathbb{R}_+; \mathbb{R}^d)$ *and* $x(t)$ *converges (as* $t \to +\infty$*) to a critical point of* f.

Remark 3.2. The boundedness assumption in assertion (ii) can be dropped if ∇f is supposed to be globally Lipschitz continuous.

Proof. (i) We will start by showing the existence of a solution. Setting $Z = (x, y)$, (3.1) can be equivalently written as

$$\dot{Z}(t) + \nabla\mathcal{G}(Z(t)) + \mathcal{D}(t, Z(t)) = 0, \quad Z(0) = (x_0, y_0), \tag{3.2}$$

where $\mathcal{G}(Z) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is the function defined by $\mathcal{G}(Z) = \beta f(x)$ and the time-dependent operator $\mathcal{D} : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ is given by:

$$\mathcal{D}(t, Z) \overset{\text{def}}{=} \left(-\left(\frac{1}{\beta} - \gamma(t)\right) x + \frac{1}{\beta} y, -\left(\frac{1}{\beta} - \gamma(t) - \beta\gamma'(t)\right) x + \frac{1}{\beta} y \right).$$

Since the map $(t, Z) \mapsto \nabla\mathcal{G}(Z) + \mathcal{D}(t, Z)$ is continuous in the first variable and locally Lipschitz in the second (by hypothesis (H_0) and the assumptions on γ), by the classical Cauchy-Lipschitz theorem, we have that there exists $T_{\max} > 0$ and a unique maximal solution of (3.2) denoted $Z \in C^1([0, T_{\max}[; \mathbb{R}^d \times \mathbb{R}^d)$. Consequently, there exists a unique maximal solution of (ISEHD) $x \in C^2([0, T_{\max}[; \mathbb{R}^d)$.

Let us consider the energy function $V : [0, T_{\max}[\to \mathbb{R}$ defined by

$$V(t) = f(x(t)) + \frac{1}{2}\|\dot{x}(t) + \beta\nabla f(x(t))\|^2.$$

We will prove that it is indeed a Lyapunov function for (ISEHD). We see that

$$\begin{aligned}
V'(t) &= \langle \nabla f(x(t)), \dot{x}(t) \rangle + \langle \ddot{x}(t) + \beta\frac{d}{dt}\nabla f(x(t)), \dot{x}(t) + \beta\nabla f(x(t)) \rangle \\
&= \langle \nabla f(x(t)), \dot{x}(t) \rangle + \langle -\gamma(t)\dot{x}(t) - \nabla f(x(t)), \dot{x}(t) + \beta\nabla f(x(t)) \rangle \\
&= -\langle \gamma(t)\dot{x}(t), \dot{x}(t) \rangle - \beta\langle \gamma(t)\dot{x}(t), \nabla f(x(t)) \rangle - \beta\|\nabla f(x(t))\|^2 \\
&\le -c\|\dot{x}(t)\|^2 + \frac{\beta^2\|\gamma(t)\dot{x}(t)\|^2}{2\varepsilon} + \frac{\varepsilon\|\nabla f(x(t))\|^2}{2} \\
&\le -c\|\dot{x}(t)\|^2 + \frac{\beta^2 C^2\|\dot{x}(t)\|^2}{2\varepsilon} + \frac{\varepsilon\|\nabla f(x(t))\|^2}{2},
\end{aligned}$$

where the last bound is due to Young's inequality with $\varepsilon > 0$. Now let $\varepsilon = \frac{\beta^2 C^2}{c}$, then

$$V'(t) \le -\frac{c}{2}\|\dot{x}(t)\|^2 - \beta\left(1 - \frac{\beta C^2}{2c}\right)\|\nabla f(x(t))\|^2 - \delta_1(\|\dot{x}(t)\|^2 + \|\nabla f(x(t))\|^2). \tag{3.3}$$

For $\delta_1 \overset{\text{def}}{=} \min\left(\frac{c}{2}, \beta\left(1 - \frac{\beta C^2}{2c}\right)\right) > 0$.

We will now show that the maximal solution Z of (3.2) is actually global. For this, we use a standard argument and argue by contradiction assuming that $T_{\max} < +\infty$. It is sufficient to prove that x and y have a limit as $t \to T_{\max}$, and local existence will contradict the maximality of $T_{\max}$. Integrating inequality (3.3), we obtain $\dot{x} \in \mathrm{L}^2([0, T_{\max}[; \mathbb{R}^d)$ and $\nabla f \circ x \in \mathrm{L}^2([0, T_{\max}[; \mathbb{R}^d)$, hence implying that $\dot{x} \in \mathrm{L}^1([0, T_{\max}[; \mathbb{R}^d)$ and $\nabla f \circ x \in \mathrm{L}^1([0, T_{\max}[; \mathbb{R}^d)$, which in turn entails that $(x(t))_{t\in[0,T_{\max}[}$ satisfies the Cauchy property whence we get that $\lim_{t\to T_{\max}} x(t)$ exists. Besides, by the first equation of (3.1), we have that $\lim_{t\to T_{\max}} y(t)$ exists if $\lim_{t\to T_{\max}} x(t), \lim_{t\to T_{\max}} \nabla f(x(t))$ and $\lim_{t\to T_{\max}} \dot{x}(t)$ exist. We have already that the first two limits exist by continuity of ∇f, and thus we just have to check that $\lim_{t\to T_{\max}} \dot{x}(t)$ exists. A sufficient condition would be to prove that $\ddot{x} \in \mathrm{L}^1([0, T_{\max}[; \mathbb{R}^d)$. By (ISEHD) this will hold if $\dot{x}, \nabla f \circ x, (\nabla^2 f \circ x)\dot{x}$ are in $\mathrm{L}^1([0, T_{\max}[; \mathbb{R}^d)$. We have already checked that the first two terms are in $\mathrm{L}^1([0, T_{\max}[; \mathbb{R}^d)$. To conclude, it remains to check that $\nabla^2 f \circ x \in \mathrm{L}^\infty([0, T_{\max}[; \mathbb{R}^d)$ and this is true since $\nabla^2 f$ is continuous, x is continuous on $[0, T_{\max}]$, and the latter is compact. Consequently, the solution Z of (3.2) is global, thus the solution x of (ISEHD) is also global.

(ii) Integrating (3.3) and using that V is well-defined for every $t > 0$ and is bounded from below, we deduce that $\dot{x} \in \mathrm{L}^2(\mathbb{R}_+; \mathbb{R}^d)$, and $\nabla f \circ x \in \mathrm{L}^2(\mathbb{R}_+; \mathbb{R}^d)$.

(iii) We recall that we are assuming that $\sup_{t\ge 0}\|x(t)\| < +\infty$ and $f \in C^2(\mathbb{R}^d)$, hence

$$\sup_{t\ge 0}\|\nabla^2 f(x(t))\| < +\infty.$$

In turn, ∇f is Lipschitz continuous on bounded sets. Moreover, as $\dot{x} \in \mathrm{L}^2(\mathbb{R}_+; \mathbb{R}^d)$ and is continuous, then $\dot{x} \in \mathrm{L}^\infty(\mathbb{R}_+; \mathbb{R}^d)$. The last two facts imply that $t \mapsto \nabla f(x(t))$ is uniformly continuous. In fact, for every $t, s \ge 0$, we have

$$\|\nabla f(x(t)) - \nabla f(x(s))\| \le \sup_{\tau\ge 0}\left\|\nabla^2 f(x(\tau))\right\| \|\dot{x}(\tau)\| \, |t - s|.$$

This combined with $\nabla f \circ x \in \mathrm{L}^2(\mathbb{R}_+; \mathbb{R}^d)$ yields

$$\lim_{t\to+\infty}\|\nabla f(x(t))\| = 0.$$

We also have that $\frac{d}{dt}\nabla f(x(t)) = \nabla^2 f(x(t))\dot{x}(t)$, and thus $(\nabla^2 f \circ x)\dot{x} \in \mathrm{L}^\infty(\mathbb{R}_+; \mathbb{R}^d)$. We also have $\nabla f \circ x \in \mathrm{L}^\infty(\mathbb{R}_+; \mathbb{R}^d)$ by continuity of ∇f and boundedness of x. It then follows from (ISEHD) that $\ddot{x} \in \mathrm{L}^\infty(\mathbb{R}_+; \mathbb{R}^d)$. This implies that

$$\|\dot{x}(t) - \dot{x}(s)\| \le \sup_{\tau\ge 0}\|\ddot{x}(\tau)\| \, |t - s|,$$

meaning that $t \mapsto \dot{x}(t)$ is uniformly continuous. Recalling that $\dot{x} \in \mathrm{L}^2(\mathbb{R}_+; \mathbb{R}^d)$ gives that

$$\lim_{t\to+\infty}\|\dot{x}(t)\| = 0.$$

We have from (3.3) that V is non-increasing. Since it is bounded from below, it has a limit, *i.e.* $\lim_{t\to+\infty} V(t)$ exists and we will denote this limit by $\tilde{L}$. Recall from the definition of V that

$$f(x(t)) = V(t) - \frac{1}{2}\|\dot{x}(t) + \beta\nabla f(x(t))\|^2.$$

Using the above three limits we get

$$\lim_{t\to+\infty} f(x(t)) = \lim_{t\to+\infty} V(t) = \tilde{L}.$$

(iv) From boundedness of $(x(t))_{t\geq 0}$, by a Lyapunov argument (see *e.g.* [28, Proposition 4.1], [75]), the set of its cluster points $\mathfrak{C}(x(\cdot))$ satisfies:

$$\begin{cases} \mathfrak{C}(x(\cdot)) \subseteq \mathrm{crit}(f); \\ \mathfrak{C}(x(\cdot)) \text{ is non-empty, compact and connected;} \\ f \text{ is constant on } \mathfrak{C}(x(\cdot)). \end{cases} \tag{3.4}$$

We consider the function

$$E : (x, v, w) \in \mathbb{R}^{3d} \mapsto f(x) + \frac{1}{2}\|v + w\|^2. \tag{3.5}$$

Since f is definable, so is E as the sum of a definable function and an algebraic one. Therefore, E satisfies the KŁ inequality [4]. Let $\mathfrak{C}_1 = \mathfrak{C}(x(\cdot)) \times \{0_d\} \times \{0_d\}$. Observe that E takes the constant value $\tilde{L}$ on $\mathfrak{C}_1$ and $\mathfrak{C}_1 \subset \mathrm{crit}(E)$. It then follows from the uniformized KŁ property [76, Lemma 6] that $\exists r, \eta > 0$ and $\exists \psi \in \kappa(0, \eta)$ such that for all $(x, v, w) \in \mathbb{R}^{3d}$ verifying $x \in \mathfrak{C}(x(\cdot)) + B_r, v \in B_r, w \in B_r$ (where B_r is the $\mathbb{R}^d$-ball centered at 0_d with radius r) and $0 < E(x, v, w) - \tilde{L} < \eta$, one has

$$\psi'(E(x, v, w) - \tilde{L})\|\nabla E(x, v, w)\| \geq 1. \tag{3.6}$$

It is clear that $V(t) = E(x(t), \dot{x}(t), \beta\nabla f(x(t)))$, and that $x^\star \in \mathrm{crit}(f)$ if and only if $(x^\star, 0, 0) \in \mathrm{crit}(E)$.

Let us define the translated Lyapunov function $\tilde{V}(t) = V(t) - \tilde{L}$. By the properties of V proved above, we have $\lim_{t\to+\infty} \tilde{V}(t) = 0$ and $\tilde{V}$ is non-increasing, and we can conclude that $\tilde{V}(t) \geq 0$ for every $t > 0$. Without loss of generality, we may assume that $\tilde{V}(t) > 0$ for every $t > 0$ (since otherwise $\tilde{V}(t)$ is eventually zero and thus $\dot{x}(t)$ is eventually zero in view of (3.3), meaning that $x(\cdot)$ has finite length). This in turn implies that $\lim_{t\to+\infty} \psi(\tilde{V}(t)) = 0$. Define the constants $\delta_2 = \max\left(4, 1 + 4\beta^2\right)$ and $\delta_3 = \frac{\delta_1}{\sqrt{\delta_2}}$. We have from (3.4) that $\lim_{t\to+\infty} \mathrm{dist}(x(t), \mathfrak{C}(x(\cdot))) = 0$. This together with the convergence claims on $\dot{x}$, $\nabla f(x)$ and $\tilde{V}$ imply that there exists $T > 0$ large enough such that for all $t \geq T$

$$\begin{cases} x(t) \in \mathfrak{C}(x(\cdot)) + B_r, \\ \|\dot{x}(t)\| < r, \\ \beta\|\nabla f(x(t))\| < r, \\ 0 < \tilde{V}(t) < \eta, \\ \frac{1}{\delta_3}\psi(\tilde{V}(t)) < \frac{r}{2\sqrt{2}}. \end{cases} \tag{3.7}$$

We are now in position to apply (3.6) to obtain

$$\psi'(\tilde{V}(t))\|\nabla E(x(t),\dot{x}(t),\beta\nabla f(x(t)))\| \geq 1, \quad \forall t \geq T. \tag{3.8}$$

On the other hand, for every $t \geq T$:

$$-\frac{d}{dt}\psi(\tilde{V}(t)) = \psi'(\tilde{V}(t))(-\tilde{V}'(t)) \geq -\frac{\tilde{V}'(t)}{\|\nabla E(x(t),\dot{x}(t),\beta\nabla f(x(t)))\|}. \tag{3.9}$$

Additionally, for every $t > 0$ we have the bounds

$$\begin{aligned} -\tilde{V}'(t) &\geq \delta_1(\|\dot{x}(t)\|^2 + \|\nabla f(x(t))\|^2), \\ \|\nabla E(x(t),\dot{x}(t),\beta\nabla f(x(t)))\|^2 &\leq \delta_2(\|\dot{x}(t)\|^2 + \|\nabla f(x(t))\|^2). \end{aligned} \tag{3.10}$$

Combining the two previous bounds, then for every $t > 0$:

$$\|\nabla E(x(t),\dot{x}(t),\beta\nabla f(x(t)))\| \leq \sqrt{\frac{\delta_2}{\delta_1}}\sqrt{-\tilde{V}'(t)}. \tag{3.11}$$

By (3.9), for every $t \in [T,+\infty[$

$$-\frac{d}{dt}\psi(\tilde{V}(t)) \geq \sqrt{\frac{\delta_1}{\delta_2}}\frac{-\tilde{V}'(t)}{\sqrt{-\tilde{V}'(t)}} = \sqrt{\frac{\delta_1}{\delta_2}}\sqrt{-\tilde{V}'(t)} \geq \delta_3\sqrt{\|\dot{x}(t)\|^2 + \|\nabla f(x(t))\|^2}. \tag{3.12}$$

Integrating from T to $+\infty$, we obtain

$$\int_T^{+\infty} \sqrt{\|\dot{x}(t)\|^2 + \|\nabla f(x(t))\|^2}dt \leq \frac{1}{\delta_3}\psi(V(T)) < \frac{r}{2\sqrt{2}}. \tag{3.13}$$

Thus

$$\int_T^{+\infty} \|\dot{x}(t)\|dt \leq \int_T^{+\infty} \sqrt{\|\dot{x}(t)\|^2 + \|\nabla f(x(t))\|^2}dt < \frac{r}{2\sqrt{2}},$$

this implies that $\dot{x} \in \mathrm{L}^1(\mathbb{R}_+;\mathbb{R}^d)$. Therefore $x(t)$ has the Cauchy property and this in turn implies that $\lim_{t\to+\infty} x(t)$ exists, and is a critical point of f since $\lim_{t\to+\infty}\|\nabla f(x(t))\| = 0$.

□

3.1.2 Trap avoidance

In the previous section, we have seen the convergence of the trajectory to a critical point of the objective, which includes strict saddle points. We will call trap avoidance the effect of avoiding such points at the limit. If the objective function satisfies the strict saddle property (recall Definition 2.5) as is the case for a Morse function, this would imply convergence to a local minimum of the objective. The following theorem gives conditions to obtain such an effect.

Theorem 3.3. *Let $c > 0$, assume that $0 < \beta < \frac{2}{c}$ and take $\gamma \equiv c$. Suppose that $f : \mathbb{R}^d \to \mathbb{R}$ satisfies (H_0) and is a Morse function. Consider (ISEHD) in this setting. If the solution trajectory x is bounded over $\mathbb{R}_+$, then the conclusions of Theorem 3.1 hold. If, moreover, $\beta \neq \frac{1}{c}$, then for Lebesgue almost all initial conditions $x_0, v_0 \in \mathbb{R}^d$, $x(t)$ converges (as $t \to +\infty$) to a local minimum of f.*

Proof. Since Morse functions are C^2 and satisfy the KŁ inequality (see Remark 2.13), then all the claims of Theorem 3.1, and in particular (iv)[3], hold.

As in [29, Theorem 4], we will use the global stable manifold theorem [77, page 223] to get the last claim. We recall that (ISEHD) is equivalent to (3.1), and that we are in the case $\gamma(t) = c$ for all t, *i.e.*,

$$\begin{cases} \dot{x}(t) + \beta\nabla f(x(t)) - \left(\frac{1}{\beta} - c\right) x(t) + \frac{1}{\beta} y(t) & = 0, \\ \dot{y}(t) - \left(\frac{1}{\beta} - c\right) x(t) + \frac{1}{\beta} y(t) & = 0, \end{cases} \tag{3.14}$$

with initial conditions $x(0) = x_0, y(0) = y_0 \stackrel{\text{def}}{=} -\beta(v_0 + \beta\nabla f(x_0)) + (1 - \beta c)x_0$. Let us consider $F : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ defined by

$$F(x, y) = \left(-\beta\nabla f(x) + \left(\frac{1}{\beta} - c\right) x - \frac{1}{\beta} y, \left(\frac{1}{\beta} - c\right) x - \frac{1}{\beta} y\right).$$

Defining $z(t) = (x(t), y(t))$ and $z_0 = (x_0, v_0) \in \mathbb{R}^{2d}$, then (3.14) is equivalent to the Cauchy problem

$$\begin{cases} \dot{z}(t) & = F(z(t)), \\ z(0) & = z_0. \end{cases} \tag{3.15}$$

We stated that when $0 < \beta < \frac{2}{c}$ and f is definable (see the first claim above), then the solution trajectory $z(t)$ converges (as $t \to +\infty$) to an equilibrium point of F. Let us denote $\Phi(z_0, t)$, the value at t of the solution (3.15) with initial condition z_0. Assume that $\hat{z}$ is a hyperbolic equilibrium point of F (to be shown below), meaning that $F(\hat{z}) = 0$ and that no eigenvalue of $J_F(\hat{z})$ has zero real part. Consider the invariant set

$$W^s(\hat{z}) = \{z_0 \in \mathbb{R}^{2d} : \lim_{t \to +\infty} \Phi(z_0, t) = \hat{z}\}.$$

The global stable manifold theorem [77, page 223] asserts that $W^s(\hat{z})$ is an immersed submanifold of $\mathbb{R}^{2d}$, whose dimension equals the number of eigenvalues of $J_F(\hat{z})$ with negative real part.

First, we will prove that each equilibrium point of F is hyperbolic. We notice that the set of equilibrium points of F is $\{(\hat{x}, (1 - \beta c)\hat{x}) : \hat{x} \in \text{crit}(f)\}$. On the other hand, we compute

$$J_F(x, y) = \begin{pmatrix} -\beta\nabla^2 f(x) + \left(\frac{1}{\beta} - c\right) I_d & -\frac{1}{\beta} I_d \\ \left(\frac{1}{\beta} - c\right) I_d & -\frac{1}{\beta} I_d \end{pmatrix}.$$

Let $\hat{z} = (\hat{x}, (1 - \beta c)\hat{x})$, where $\hat{x} \in \text{crit}(f)$. Then the eigenvalues of $J_F(\hat{z})$ are characterized by the roots in $\lambda \in \mathbb{C}$ of

$$\det\left(\begin{pmatrix} -\beta\nabla^2 f(\hat{x}) + \left(\frac{1}{\beta} - c - \lambda\right) I_d & -\frac{1}{\beta} I_d \\ \left(\frac{1}{\beta} - c\right) I_d & -\left(\lambda + \frac{1}{\beta}\right) I_d \end{pmatrix}\right) = 0. \tag{3.16}$$

By Lemma 2.1, we have that (3.16) is equivalent to

$$\det((1 + \lambda\beta)\nabla^2 f(\hat{x}) + (\lambda^2 + \lambda c)I_d) = 0. \tag{3.17}$$

[3] In fact, the proof is even more straightforward since the set of cluster points $\mathfrak{C}(x(\cdot))$ satisfies (3.4) and the critical points are isolated; see the proof of [28, Theorem 4.1].

If $\lambda = -\frac{1}{\beta}$, then by (3.17), $\beta c = 1$, which is excluded by hypothesis. Therefore, $-\frac{1}{\beta}$ cannot be an eigenvalue, *i.e.* $\lambda \neq -\frac{1}{\beta}$. We then obtain that (3.17) is equivalent to

$$\det\left(\nabla^2 f(\hat{x}) + \frac{\lambda^2 + \lambda c}{(1+\lambda\beta)} I_d\right) = 0. \tag{3.18}$$

It follows that λ satisfies (3.18) if and only if

$$\frac{\lambda^2 + \lambda c}{(1+\lambda\beta)} = -\eta$$

where $\eta \in \mathbb{R}$ is an eigenvalue of $\nabla^2 f(\hat{x})$. Equivalently,

$$\lambda^2 + (c + \eta\beta)\lambda + \eta = 0. \tag{3.19}$$

Let $\Delta_\lambda \stackrel{\text{def}}{=} (c+\eta\beta)^2 - 4\eta$. We distinguish two cases.

- $\Delta_\lambda \geq 0$: then the roots of (3.19) are real and we rewrite (3.19) as
$$\lambda(\lambda + (c + \eta\beta)) = -\eta,$$
since $\eta \neq 0$ (because f is a Morse function), then $\lambda \neq 0$.
- $\Delta_\lambda < 0$: then (3.19) has a pair of complex conjugate roots whose real part is $-\frac{c+\eta\beta}{2}$. Besides, $\Delta_\lambda = c^2 + 2\beta\eta c + \eta^2\beta^2 - 4\eta$ can be seen as a quadratic on c whose discriminant is given by $\Delta_c = 16\eta$. The fact that $\Delta_\lambda < 0$ implies $\Delta_c > 0$, and thus $\eta > 0$, therefore $-\frac{c+\eta\beta}{2} < 0$.

Overall, this shows that every equilibrium point of F is hyperbolic.

Let us recall that $\text{crit}(f) = \bigcup_{k\in I}\{\hat{x}_k\}$. Thus, the set of equilibria of F is also finite and each one takes the form $\hat{z}_k = (\hat{x}_k, (1-\beta c)\hat{x}_k)$. Since we have already shown that each solution trajectory x of (ISEHD) converges towards some $\hat{x}_k$, the following partition then holds

$$\mathbb{R}^d \times \mathbb{R}^d = \bigcup_{k\in I} W^s(\hat{z}_k).$$

Let

$$I^- = \{k \in I : \text{ each eigenvalue of } J_F(\hat{z}_k) \text{ has negative real part.}\},$$

and $J \stackrel{\text{def}}{=} I \setminus I^-$. Now, the global stable manifold theorem [77, page 223] allows to claim that $W^s(\hat{z}_k)$ is an immersed submanifold of $\mathbb{R}^{2d}$ whose dimension is $2d$ when $k \in I^-$ and at most $2d-1$ when $k \in J$.

Let $k \in I^-$, we claim that $\nabla^2 f(\hat{x}_k)$ has only positive eigenvalues. By contradiction, let us assume that $\eta_0 < 0$ is an eigenvalue of $\nabla^2 f(\hat{x}_k)$ ($\eta_0 = 0$ is not possible due to the Morse hypothesis). Each solution λ of (3.19) is an eigenvalue of $J_F(\hat{z}_k)$ and one of these solutions is

$$\frac{-(c+\eta\beta) + \sqrt{(c+\eta\beta)^2 - 4\eta_0}}{2}$$

which is positive since $\eta_0 < 0$. We then have

$$\frac{-(c+\eta_0\beta) + \sqrt{(c+\eta_0\beta)^2 - 4\eta_0}}{2} > \frac{-(c+\eta_0\beta) + |c+\eta_0\beta|}{2} \geq 0,$$

hence contradicting the assumption that $k \in I^-$. In conclusion, the set of initial conditions z_0 such that $\Phi(z_0, t)$ converges to $(x_b, (1-\beta c)x_b)$ (as $t \to +\infty$), where x_b is not a local minimum of f is $\bigcup_{k \in J} W^s(\hat{z}_k)$ which has Lebesgue measure zero. Therefore, due to the equivalence between (3.1) and (ISEHD) and that Morse functions satisfy the strict saddle property (see Remark 2.7), we indeed have that for almost all initial conditions $x_0, v_0 \in \mathbb{R}^d$, the solution trajectory of (ISEHD) will converge to a local minimum of f. □

3.1.3 Convergence rate

When the objective function is definable, we now provide the convergence rate on the Lyapunov function E in (3.5), hence f, and on the solution trajectory x.

Theorem 3.4. *Consider the setting of Theorem 3.1 with f being also definable and the solution trajectory x is bounded. Recall the function E from (3.5), which is also definable, and denote ψ its desingularizing function and Ψ any primitive of $-\psi'^2$. Then, $x(t)$ converges (as $t \to +\infty$) to $x_\infty \in \mathrm{crit}(f)$. Denote $\tilde{V}(t) \stackrel{\text{def}}{=} E(x(t), \dot{x}(t), \beta\nabla f(x(t))) - f(x_\infty)$. The following rates of convergence hold:*

- *If $\lim_{t \to 0} \Psi(t) \in \mathbb{R}$, we have $E(x(t), \dot{x}(t), \beta\nabla f(x(t)))$ converges to $f(x_\infty)$ in finite time.*
- *If $\lim_{t \to 0} \Psi(t) = +\infty$, there exists some $t_1 \geq 0$ such that*

$$\tilde{V}(t) = \mathcal{O}(\Psi^{-1}(t - t_1)). \tag{3.20}$$

Moreover,

$$\|x(t) - x_\infty\| = \mathcal{O}(\psi \circ \Psi^{-1}(t - t_1)). \tag{3.21}$$

Proof. This proof is a generalization of [5, Theorem 2.7] to the dynamics (ISEHD). Let $\delta_0 \stackrel{\text{def}}{=} \frac{\delta_2}{\delta_1}, \delta_3 > 0$ and $T > 0$ for $\delta_1, \delta_2, \delta_3, T$ defined in the proof of Theorem 3.1. Using (3.10) then (3.9), we have for $t > T$

$$\begin{aligned}
\frac{d}{dt}\Psi(\tilde{V}(t)) &= \Psi'(\tilde{V}(t))\tilde{V}'(t) \\
&= -\psi'^2(\tilde{V}(t))\tilde{V}'(t) \\
&\geq \delta_0 \psi'^2(\tilde{V}(t))\|\nabla E(x(t), \dot{x}(t), \beta\nabla f(x(t))\|^2 \\
&\geq \delta_0.
\end{aligned} \tag{3.22}$$

Integrating on both sides from T to t we obtain that for every $t > T$

$$\Psi(\tilde{V}(t)) \geq \delta_0(t - T) + \Psi(\tilde{V}(T)).$$

Following the arguments shown in [78, Theorem 3.1.12], if $\lim_{t \to 0} \Psi(t) \in \mathbb{R}$, then $\tilde{V}(t)$ converges to 0 in finite time. Otherwise, we take the inverse of Ψ, which is non-increasing, on both sides of (3.22) to obtain the desired bound. Finally, using (3.13) we also have for every $t > T$

$$\|x(t) - x_\infty\| \leq \int_t^{+\infty} \|\dot{x}(s)\| ds \leq \frac{1}{\delta_3}\psi(\tilde{V}(t)) \leq \frac{1}{\delta_3}\psi \circ \Psi^{-1}(\delta_0(t - T) + \Psi(\tilde{V}(T))). \tag{3.23}$$

□

Remark 3.5. Observe that the convergence rate (3.20) holds also on $f(x(t)) - f(x_\infty)$ and $\|\dot{x}(t) + \nabla f(x(t))\|^2$.

We now specialize this to the Łojasiewicz case.

Corollary 3.6. *Consider the setting of Theorem 3.4 where now f satisfies the Łojasiewicz inequality with desingularizing function $\psi_f(s) = c_f s^{1-q}$, $q \in [0, 1[$, $c_f > 0$. Then there exists some $t_1 > 0$ such that the the following convergence rates hold:*

- *If $q \in [0, \frac{1}{2}]$, then*

$$\tilde{V}(t) = \mathcal{O}\left(\exp(-(t-t_1))\right) \quad \text{and} \quad \|x(t) - x_\infty\| = \mathcal{O}\left(\exp\left(\frac{t-t_1}{2}\right)\right). \tag{3.24}$$

- *If $q \in]\frac{1}{2}, 1[$, then*

$$\tilde{V}(t) = \mathcal{O}\left((t-t_1)^{\frac{-1}{2q-1}}\right) \quad \text{and} \quad \|x(t) - x_\infty\| = \mathcal{O}\left((t-t_1)^{-\frac{1-q}{2q-1}}\right). \tag{3.25}$$

Proof. E is a separable quadratic perturbation of f. But a quadratic function is Łojasiewicz with exponent $1/2$. It then follows from the Łojasiewicz exponent calculus rule in [79, Theorem 3.3] that the desingularizing function of E is $\psi_E(s) = c_E s^{1-q_E}$ for some $c_E > 0$ and $q_E = \max\left(q, \frac{1}{2}\right)$. Then,

- If $q \in [0, \frac{1}{2}]$ then $q_E = \frac{1}{2}$ and $\Psi(s) = \frac{c_1^2}{4} \ln\left(\frac{1}{s}\right)$. This implies that $\Psi^{-1}(s) = \frac{4}{c_1^2} \exp(-s)$.
- If $q \in]\frac{1}{2}, 1[$ then $q_E = q$ and $\Psi(s) = \frac{c_1^2}{4(2q-1)} s^{1-2q}$. This implies that $\Psi^{-1}(s) = \frac{4(2q-1)}{c_1^2} s^{\frac{-1}{2q-1}}$.

We conclude in both cases by using Theorem 3.4. □

3.2 Algorithmic scheme

Now we will consider the following finite differences explicit discretization of (ISEHD) with step-size $h > 0$ and for $k \geq 1$:

$$\frac{x_{k+1} - 2x_k + x_{k-1}}{h^2} + \gamma(kh)\frac{x_{k+1} - x_k}{h} + \beta\frac{\nabla f(x_k) - \nabla f(x_{k-1})}{h} + \nabla f(x_k) = 0.$$

Rearranging, this equivalently reads

$$\begin{cases} y_k &= x_k + \alpha_k(x_k - x_{k-1}) - \beta_k(\nabla f(x_k) - \nabla f(x_{k-1})), \\ x_{k+1} &= y_k - s_k \nabla f(x_k), \end{cases} \tag{ISEHD-Disc}$$

with initial conditions $x_0, x_1 \in \mathbb{R}^d$, where $\alpha_k \stackrel{\text{def}}{=} \frac{1}{1+\gamma_k h}$, $\gamma_k \stackrel{\text{def}}{=} \gamma(kh)$, $\beta_k \stackrel{\text{def}}{=} \beta h \alpha_k$, $s_k \stackrel{\text{def}}{=} h^2 \alpha_k$

3.2.1 Global convergence and trap avoidance

The following theorem summarizes our main results on the behavior of (ISEHD-Disc). Observe that as the discretization is explicit, we will need ∇f to be globally Lipschitz continuous.

Theorem 3.7. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be satisfying (H_0) with ∇f being globally L-Lipschitz-continuous. Consider the scheme (ISEHD-Disc) with $h > 0$, $\beta \geq 0$ and $c \leq \gamma_k \leq C$ for some $c, C > 0$ and all $k \in \mathbb{N}$. Then the following holds:*

(i) If $\beta + \frac{h}{2} < \frac{c}{L}$, then $(\|\nabla f(x_k)\|)_{k\in\mathbb{N}} \in \ell^2(\mathbb{N})$, and $(\|x_{k+1} - x_k\|)_{k\in\mathbb{N}} \in \ell^2(\mathbb{N})$, in particular

$$\lim_{k\to+\infty} \|\nabla f(x_k)\| = 0.$$

(ii) Moreover, if $(x_k)_{k\in\mathbb{N}}$ is bounded and f is definable, then $(\|x_{k+1} - x_k\|)_{k\in\mathbb{N}} \in \ell^1(\mathbb{N})$ and x_k converges (as $k \to +\infty$) to a critical point of f.

(iii) Furthermore, if $\gamma_k \equiv c > 0$, $0 < \beta < \frac{c}{L}, \beta \neq \frac{1}{c}$, and $h < \min(2\left(\frac{c}{L} - \beta\right), \frac{1}{L\beta})$, then for almost all $x_0, x_1 \in \mathbb{R}^d$, x_k converges (as $k \to +\infty$) to a critical point of f that is not a strict saddle. Consequently, if f satisfies the strict saddle property then for almost all $x_0, x_1 \in \mathbb{R}^d$, x_k converges (as $k \to +\infty$) to a local minimum of f.

Remark 3.8. When $\beta = 0$, we recover the HBF method and the condition $h < \min(2\left(\frac{c}{L} - \beta\right), \frac{1}{L\beta})$ becomes $h < \frac{2c}{L}$.

Proof. (i) By definition of x_{k+1} in (ISEHD-Disc), for $k \in \mathbb{N}^*$

$$x_{k+1} = \operatorname*{argmin}_{x \in \mathbb{R}^d} \frac{1}{2}\|x - (y_k - s_k \nabla f(x_k))\|^2. \tag{3.26}$$

$1-$strong convexity of $x \mapsto \frac{1}{2}\|x - (y_k - s_k \nabla f(x_k))\|^2$ then yields

$$\frac{1}{2}\|x_{k+1} - (y_k - s_k \nabla f(x_k))\|^2 \leq \frac{1}{2}\|x_k - (y_k - s_k \nabla f(x_k))\|^2 - \frac{1}{2}\|x_{k+1} - x_k\|^2. \tag{3.27}$$

Let $\underline{\alpha} = \frac{1}{1+Ch}, \bar{\alpha} = \frac{1}{1+ch}, \underline{s} = h^2\underline{\alpha}, \bar{s} = h^2\bar{\alpha}$, and thus for every $k \in \mathbb{N}$, $\underline{\alpha} \leq \alpha_k \leq \bar{\alpha}$ and $\underline{s} \leq s_k \leq \bar{s}$. Let also $v_k \stackrel{\text{def}}{=} x_k - x_{k-1}$, $z_k \stackrel{\text{def}}{=} \alpha_k v_k - \beta_k(\nabla f(x_k) - \nabla f(x_{k-1}))$, then $y_k = x_k + z_k$. After expanding the terms of (3.27) we have that

$$\begin{aligned} \langle \nabla f(x_k), v_{k+1}\rangle &\leq -\frac{\|v_{k+1}\|^2}{s_k} + \frac{1}{s_k}\langle v_{k+1}, z_k\rangle \\ &\leq -\frac{\|v_{k+1}\|^2}{\bar{s}} + \frac{1}{h^2}\langle v_{k+1}, v_k\rangle - \frac{\beta}{h}\langle v_{k+1}, \nabla f(x_k) - \nabla f(x_{k-1})\rangle. \end{aligned} \tag{3.28}$$

By the descent lemma for L-smooth functions, we obtain

$$f(x_{k+1}) \leq f(x_k) + \langle \nabla f(x_k), v_{k+1}\rangle + \frac{L}{2}\|v_{k+1}\|^2. \tag{3.29}$$

Using the bound in (3.28), we get

$$f(x_{k+1}) \leq f(x_k) + \frac{1}{h^2}\langle v_{k+1}, v_k\rangle - \frac{\beta}{h}\langle v_{k+1}, \nabla f(x_k) - \nabla f(x_{k-1})\rangle - \left(\frac{1}{\bar{s}} - \frac{L}{2}\right)\|v_{k+1}\|^2. \tag{3.30}$$

According to our hypothesis $h < \frac{2c}{L}$, so $h < \frac{c+\sqrt{c^2+2L}}{L}$ and this implies that $\bar{s} < \frac{2}{L}$. Using Young's inequality twice, for $\varepsilon, \varepsilon' > 0$, the fact that ∇f is $L-$Lipschitz, and adding $\frac{\varepsilon+\varepsilon'}{2}\|v_{k+1}\|^2$ at both sides, then

$$\begin{aligned} f(x_{k+1}) + \frac{\varepsilon+\varepsilon'}{2}\|v_{k+1}\|^2 &\leq f(x_k) + \frac{\varepsilon+\varepsilon'}{2}\|v_k\|^2 \\ &\quad + \left(\frac{1}{2}\left[\frac{1}{h^4\varepsilon} + \varepsilon + \frac{\beta^2L^2}{h^2\varepsilon'} + \varepsilon'\right] - \left(\frac{1}{\bar{s}} - \frac{L}{2}\right)\right)\|v_{k+1}\|^2. \end{aligned} \tag{3.31}$$

In order to make the last term negative, we want to impose

$$\frac{1}{2}\left[\frac{1}{h^4\varepsilon} + \varepsilon + \frac{\beta^2L^2}{h^2\varepsilon'} + \varepsilon'\right] < \left(\frac{1}{\bar{s}} - \frac{L}{2}\right). \tag{3.32}$$

Minimizing the left-hand side with respect to $\varepsilon, \varepsilon' > 0$ we get $\varepsilon = \frac{1}{h^2}, \varepsilon' = \frac{\beta L}{h}$, and one can check that in this case, the condition (3.32) becomes equivalent to $\beta + \frac{h}{2} < \frac{c}{L}$ which is assumed in the hypothesis.

Setting $C_1 \stackrel{\text{def}}{=} \frac{1}{h^2} + \frac{\beta L}{h}, \delta \stackrel{\text{def}}{=} \frac{1}{\bar{s}} - \frac{L}{2} - \frac{1}{h^2} - \frac{\beta L}{h} > 0$, and defining $V_k \stackrel{\text{def}}{=} f(x_k) + \frac{C_1}{2}\|v_k\|^2$, we have for any $k \in \mathbb{N}^*$

$$V_{k+1} \leq V_k - \delta\|v_{k+1}\|^2. \tag{3.33}$$

Clearly, V_k is non-increasing and bounded from below, hence $\lim_{k\to+\infty} V_k$ exists (say $\tilde{L}$). Summing this inequality over k, we have that $(\|v_{k+1}\|)_{k\in\mathbb{N}} \in \ell^2(\mathbb{N})$ entailing that $\lim_{k\to+\infty}\|v_k\| = 0$. In turn,we have that $\lim_{k\to+\infty} f(x_k) = \tilde{L}$. Embarking again from the update in (ISEHD-Disc), we have

$$\begin{aligned} s_k\|\nabla f(x_k)\| = \|x_{k+1} - y_k\| &\leq \|x_{k+1} - x_k\| + \|x_k - y_k\| \\ &\leq \|v_{k+1}\| + (\alpha_k + \beta_k L)\|v_k\| \\ &\leq \|v_{k+1}\| + \|v_k\|, \end{aligned}$$

since $\bar{\alpha}(1 + \beta h L) < 1$ by hypothesis. Therefore

$$\|\nabla f(x_k)\|^2 \leq \delta_2(\|v_{k+1}\|^2 + \|v_k\|^2),$$

where $\delta_2 = \frac{2}{\underline{s}^2}$. Consequently $(\|\nabla f(x_k)\|)_{k\in\mathbb{N}} \in \ell^2(\mathbb{N})$, which implies that $\lim_{k\to+\infty}\|\nabla f(x_k)\| = 0$.

(ii) If, moreover, $(x_k)_{k\in\mathbb{N}}$ is bounded, then the set of its cluster points $\mathfrak{C}((x_k)_{k\in\mathbb{N}})$ satisfies (see *e.g.* [76, Lemma 5]):

$$\begin{cases} \mathfrak{C}((x_k)_{k\in\mathbb{N}}) \subseteq \operatorname{crit}(f); \\ \mathfrak{C}((x_k)_{k\in\mathbb{N}}) \text{ is non-empty, compact and connected;} \\ f \text{ is constant on } \mathfrak{C}((x_k)_{k\in\mathbb{N}}). \end{cases} \tag{3.34}$$

Define

$$E : (x, v) \in \mathbb{R}^{2d} \mapsto f(x) + \frac{C_1}{2}\|v\|^2. \tag{3.35}$$

Since f is definable, so is E as the sum of a definable function and an algebraic one, whence E satisfies the KŁ inequality. Let $\mathfrak{C}_1 = \mathfrak{C}((x_k)_{k\in\mathbb{N}}) \times \{0_d\}$. Since $E\big|_{\mathfrak{C}_1} = \tilde{L}, \nabla E\big|_{\mathfrak{C}_1} = 0$, $\exists r, \eta > 0, \exists \psi \in \kappa(0, \eta)$ such that for every (x, v) such that $x \in \mathfrak{C}((x_k)_{k\in\mathbb{N}}) + B_r, v \in B_r$, (where B_r is the $\mathbb{R}^d$-ball centred at 0_d with radius r) and $0 < E(x, v) - \tilde{L} < \eta$, one has

$$\psi'(E(x, v) - \tilde{L})\|\nabla E(x, v)\| \geq 1. \tag{3.36}$$

Let us define $\tilde{V}_k = V_k - \tilde{L}$, or equivalently $\tilde{V}_k = E(x_k, v_k) - \tilde{L}$. From (3.33), $\left(\tilde{V}_k\right)_{k\in\mathbb{N}}$ is a non-increasing sequence and its limit is 0 by definition of $\tilde{L}$. This implies that that $\tilde{V}_k \geq 0$ for all $k \in \mathbb{N}^*$. We may assume without loss of generality that $\tilde{V}_k > 0$. Indeed, suppose there exists $K \in \mathbb{N}$ such that $\tilde{V}_K = 0$, then the decreasing property (3.33) implies that $\tilde{V}_k = 0$ holds for all $k \geq K$. Thus $v_{k+1} = 0$, or equivalently $x_k = x_K$, for all $k \geq K$, hence $(x_k)_{k\in\mathbb{N}}$ has finite length.

Since $\lim_{k\to+\infty} \operatorname{dist}(x_k, \mathfrak{C}((x_k)_{k\in\mathbb{N}})) = 0, \lim_{k\to+\infty}\|v_k\| = 0$, and $\lim_{k\to+\infty}\tilde{V}_k = 0$, there exists $\tilde{K} \in \mathbb{N}$ such that for all $k \geq \tilde{K}$,

$$\begin{cases} x_k \in \mathfrak{C}((x_k)_{k\in\mathbb{N}}) + B_r; \\ \|v_k\| < r; \\ 0 < \tilde{V}_k < \eta. \end{cases}$$

Then, by (3.36), we have

$$\psi'(\tilde{V}_k)\|\nabla E\left(x_k, v_k\right)\| \geq 1, \quad \forall k \geq \tilde{K}. \tag{3.37}$$

By concavity of ψ and (3.37), we have

$$\begin{aligned}\psi(\tilde{V}_k) - \psi(\tilde{V}_{k+1}) &\geq -\psi'(\tilde{V}_k)(\tilde{V}_{k+1} - \tilde{V}_k) \\ &\geq \delta\psi'(\tilde{V}_k)\|v_{k+1}\|^2 \\ &\geq \delta\frac{\|v_{k+1}\|^2}{\|\nabla E\left(x_k, v_k\right)\|}.\end{aligned}$$

On the other hand,

$$\|\nabla E\left(x_k, v_k\right)\| \leq \delta_3(\|v_{k+1}\| + \|v_k\|) \tag{3.38}$$

where $\delta_3 = \sqrt{C_1^2 + \delta_2}$. Let us define for $k \in \mathbb{N}^*$, $(\Delta\psi)_k \stackrel{\text{def}}{=} \psi(\tilde{V}_k) - \psi(\tilde{V}_{k+1})$ and $\delta_4 = \frac{\delta_3}{\delta}$. We then have for all $k \geq \tilde{K}$,

$$\|v_{k+1}\|^2 \leq \delta_4(\Delta\psi)_k(\|v_k\| + \|v_{k+1}\|).$$

Using Young's inequality and concavity of $\sqrt{\cdot}$, this implies that for every $\varepsilon > 0$

$$\|v_{k+1}\| \leq \delta_4\frac{(\Delta\psi)_k}{\sqrt{2}\varepsilon} + \varepsilon\frac{\|v_{k+1}\| + \|v_k\|}{\sqrt{2}}.$$

Rearranging the terms and imposing $0 < \varepsilon < \sqrt{2}$ gives

$$\left(1 - \frac{\varepsilon}{\sqrt{2}}\right)\|v_{k+1}\| \leq \delta_4\frac{(\Delta\psi)_k}{\sqrt{2}\varepsilon} + \varepsilon\frac{\|v_k\|}{\sqrt{2}}.$$

Dividing by $\left(1 - \frac{\varepsilon}{\sqrt{2}}\right)$ on both sides, we get

$$\|v_{k+1}\| \leq \delta_4\frac{(\Delta\psi)_k}{\varepsilon(\sqrt{2} - \varepsilon)} + \varepsilon\frac{\|v_k\|}{\sqrt{2} - \varepsilon}. \tag{3.39}$$

Choosing now ε such that $0 < \varepsilon < \frac{\sqrt{2}}{2}$, we get that $0 < \frac{\varepsilon}{\sqrt{2}-\varepsilon} < 1$. Since $(\Delta\psi)_k \in \ell^1(\mathbb{N}^*)$ as a telescopic sum, we conclude that $(\|v_k\|)_{k\in\mathbb{N}} \in \ell^1(\mathbb{N}^*)$. This means that $(x_k)_{k\in\mathbb{N}}$ has finite length, hence is a Cauchy sequence, entailing that x_k has a limit (as $k \to +\infty$) denoted x_∞ which is a critical point of f since $\lim_{k\to+\infty}\|\nabla f(x_k)\| = 0$.

(iii) If $\gamma_k \equiv c$, we denote $\alpha_k \equiv \alpha = \frac{1}{1+ch}$, $\beta_k \equiv \tilde{\beta} = \beta h\alpha$, $s_k \equiv s = h^2\alpha$. Let $z_k = (x_k, x_{k-1})$ for $k \geq 1$, and $g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ defined by

$$g : (x_+, x_-) \mapsto [(1+\alpha)x_+ - \alpha x_- - (\tilde{\beta} + s)\nabla f(x_+) + \tilde{\beta}\nabla f(x_-), x_+].$$

(ISEHD-Disc) is then equivalent to

$$z_{k+1} = g(z_k). \tag{3.40}$$

To complete the proof, we will capitalize on [66, Corollary 1] which builds on the center stable manifold theorem [58, Theorem III.7]. For this, one needs to check two conditions:

(a) $\det(J_g(x_+, x_-)) \neq 0$ for every $x_+, x_- \in \mathbb{R}^d$.

(b) Let $\mathcal{A}_g^\star \overset{\text{def}}{=} \{(x,x) \in \mathbb{R}^{2d} : x \in \mathrm{crit}(f), \max_i |\lambda_i(J_g(x,x))| > 1\}$, $\mathcal{X}^\star$ be the set of strict saddle points of f, and $\hat{\mathcal{X}} \overset{\text{def}}{=} \{(x,x) \in \mathbb{R}^{2d} : x \in \mathcal{X}^\star\}$. One needs to check that $\hat{\mathcal{X}} \subset \mathcal{A}_g^\star$.

$\mathcal{A}_g^\star$ is the set of unstable fixed points. Indeed, the fixed points of g are of the form $(x^\star, x^\star)$ where $x^\star \in \mathrm{crit}(f)$.

We first compute $J_g(x_+, x_-)$, given by

$$\begin{pmatrix} (1+\alpha)I_d - (\tilde{\beta}+s)\nabla^2 f(x_+) & -\alpha I_d + \tilde{\beta}\nabla^2 f(x_-) \\ I_d & 0_{d\times d} \end{pmatrix} \tag{3.41}$$

This is a block matrix that comes in a form amenable to applying Lemma 2.1. We then have

$$\det(J_g(x_+, x_-)) = \det(\alpha I_d - \tilde{\beta}\nabla^2 f(x_-)).$$

Since the eigenvalues of $\nabla^2 f(x_-)$ are contained in $[-L, L]$, if $\alpha > L\tilde{\beta}$, then $\alpha - \tilde{\beta}\eta \neq 0$ for every eigenvalue $\eta \in \mathbb{R}$ of $\nabla^2 f(x_-)$. This implies that the first condition is satisfied, *i.e.* $\det(J_g(x_+, x_-)) \neq 0$ for every $x_+, x_- \in \mathbb{R}^d$. The condition $\alpha > L\tilde{\beta}$ in terms of h reads $h < \frac{1}{L\beta}$, since we already needed $h < 2\left(\frac{c}{L} - \beta\right)$, we just ask h to be less than the minimum of the two quantities.

To check the second condition, let us take x a strict saddle point of f, *i.e.* $x \in \mathrm{crit}(f)$ and $\lambda_{\min}(\nabla^2 f(x)) = -\eta < 0$. To compute the eigenvalues of $J_g(x,x)$ we consider

$$\det\left(\begin{pmatrix} (1+\alpha-\lambda)I_d - (\tilde{\beta}+s)\nabla^2 f(x) & -\alpha I_d + \tilde{\beta}\nabla^2 f(x) \\ I_d & -\lambda I_d \end{pmatrix}\right) = 0.$$

Again by Lemma 2.1, we get that

$$\begin{aligned}
&\det\left(\begin{pmatrix} (1+\alpha-\lambda)I_d - (\tilde{\beta}+s)\nabla^2 f(x) & -\alpha I_d + \tilde{\beta}\nabla^2 f(x) \\ I_d & -\lambda I_d \end{pmatrix}\right) = \\
&\det[(-\lambda(1+\alpha)+\lambda^2)I_d + \lambda(\tilde{\beta}+s)\nabla^2 f(x) + \alpha I_d - \tilde{\beta}\nabla^2 f(x)] = \\
&\det[(\lambda(\tilde{\beta}+s) - \tilde{\beta})\nabla^2 f(x) + (\lambda^2 - \lambda(1+\alpha)+\alpha)I_d].
\end{aligned}$$

We then need to solve for λ

$$\det[(\lambda(\tilde{\beta}+s) - \tilde{\beta})\nabla^2 f(x) + (\lambda^2 - \lambda(1+\alpha)+\alpha)I_d] = 0. \tag{3.42}$$

If $\lambda = \frac{\tilde{\beta}}{\tilde{\beta}+s}$, then (3.42) becomes

$$\left(\frac{\tilde{\beta}}{\tilde{\beta}+s}\right)^2 - \left(\frac{\tilde{\beta}}{\tilde{\beta}+s}\right)(1+\alpha) + \alpha = 0.$$

This implies that $\alpha = \frac{\tilde{\beta}}{\tilde{\beta}+s}$, which in terms of β and c is equivalent to $\beta c = 1$. But this case is excluded by hypothesis. Now we can focus on the case where $\lambda \neq \frac{\tilde{\beta}}{\tilde{\beta}+s}$ and we can rewrite (3.42) as

$$\det\left(\nabla^2 f(x) - \frac{\lambda^2 - \lambda(1+\alpha)+\alpha}{\tilde{\beta} - \lambda(\tilde{\beta}+s)} I_d\right) = 0. \tag{3.43}$$

Therefore, as argued for the time-continuous dynamic, for every eigenvalue $\eta' \in \mathbb{R}$ of $\nabla^2 f(x)$, λ satisfies (3.43) if and only if

$$\frac{\lambda^2 - \lambda(1+\alpha) + \alpha}{\tilde{\beta} - \lambda(\tilde{\beta} + s)} = \eta'.$$

where $\eta' \in \mathbb{R}$ is an eigenvalue of $\nabla^2 f(\hat{x})$. Thus if $\eta' = -\eta$ is negative, we have

$$\lambda^2 - \lambda((1+\alpha) + \eta(\tilde{\beta} + s)) + \alpha + \eta\tilde{\beta} = 0. \tag{3.44}$$

We analyze its discriminant $\Delta_\lambda = ((1+\alpha) + \eta(\tilde{\beta}+s))^2 - 4(\alpha + \eta\tilde{\beta})$. After developing the terms we get that

$$\Delta_\lambda = \alpha^2 + 2\alpha(\eta(\tilde{\beta}+s) - 1) + (\eta(\tilde{\beta}+s) + 1)^2 - 4\eta\tilde{\beta},$$

which can be seen as a quadratic equation on α. We get that its discriminant Δ_α is $-16\eta s$, which is negative (since η, s are positive), thus the quadratic equation on α does not have real roots, implying that $\Delta_\lambda > 0$. We can write the solutions of (3.44),

$$\lambda = \frac{((1+\alpha) + \eta(\tilde{\beta}+s)) \pm \sqrt{\Delta_\lambda}}{2}.$$

Let us consider the biggest solution (the one with the plus sign) and let us see that $\lambda > 1$, this is equivalent to

$$\frac{((1+\alpha) + \eta(\tilde{\beta}+s)) + \sqrt{\Delta_\lambda}}{2} > 1,$$

which in turn is equivalent to

$$\sqrt{\Delta_\lambda} > 2 - (1+\alpha) - \eta(\tilde{\beta}+s).$$

Squaring both sides of this inequality, we have

$$\begin{aligned}[(1-\alpha) - \eta(\tilde{\beta}+s)]^2 &< \Delta_\lambda \\ &= [(1+\alpha) + \eta(\tilde{\beta}+s)]^2 - 4(\alpha + \eta\tilde{\beta}).\end{aligned}$$

After expanding the terms, we see that the inequality is equivalent to $0 < 4\eta s$, which is always true as $\eta > 0$. Consequently, $\lambda > 1$ and in turn $\hat{\mathcal{X}} \subset \mathcal{A}_g^\star$.
We have then checked the two conditions (a)-(b) above. This entails that the invariant set $\{z_1 \in \mathbb{R}^{2d} : \lim_{k\to+\infty} g^k(z_1) \in \hat{\mathcal{X}}\}$ has Lebesgue measure zero. Equivalently, the set of initializations $x_0, x_1 \in \mathbb{R}^d$ for which x_k converges to a strict saddle point of f has Lebesgue measure zero.

□

3.2.2 Convergence rate

The following result provides the convergence rates for algorithm (ISEHD-Disc) in the case where f has the Łojasiewicz property. The original idea of proof for descent-like algorithms can be found in [80, Theorem 5].

Theorem 3.9. *Consider the setting of Theorem 3.7, where f also satisfies the Łojasiewicz property with exponent $q \in [0, 1[$. Then $x_k \to x_\infty \in \mathrm{crit}(f)$ as $k \to +\infty$ at the rates:*

- *If $q \in [0, \frac{1}{2}]$ then there exists $\rho \in]0, 1[$ such that*

$$\|x_k - x_\infty\| = \mathcal{O}(\rho^k). \tag{3.45}$$

- *If* $q \in]\frac{1}{2}, 1[$ *then*

$$\|x_k - x_\infty\| = \mathcal{O}\left(k^{-\frac{1-q}{2q-1}}\right). \tag{3.46}$$

Proof. Recall the function E from (3.35). Since f satisfies the Łojasiewicz property with exponent $q \in [0, 1[$, and E is a separable quadratic perturbation of f, it follows from [79, Theorem 3.3] that E has the Łojasiewicz property with exponent $q_E = \max(q, 1/2) \in [1/2, 1[$, *i.e.* there exists $c_E > 0$ such that the desingularizing function of E is $\psi_E(s) = c_E s^{1-q_E}$.

Let $v_k = x_k - x_{k-1}$ and $\Delta_k = \sum_{p=k}^{+\infty} \|v_{p+1}\|$. The triangle inequality yields $\Delta_k \geq \|x_k - x_\infty\|$ so it suffices to analyze the behavior of Δ_k to obtain convergence rates for the trajectory. Recall the constants $\delta, \delta_3, \delta_4 > 0$ and the sequences $((\Delta\psi)_k)_{k\in\mathbb{N}}$ and $\left(\tilde{V}_k\right)_{k\in\mathbb{N}}$ defined in the proof of Theorem 3.7. Denote $\lambda = \frac{\varepsilon}{\sqrt{2}-\varepsilon} \in (0, 1)$ and $M = \frac{\delta_4}{\varepsilon(\sqrt{2}-\varepsilon)}$ for $0 < \varepsilon < \frac{\sqrt{2}}{2}$. Using (3.39), we have that there exists $\tilde{K} \in \mathbb{N}$ large enough such that for all $k \geq \tilde{K}$

$$\|v_{k+1}\| \leq \lambda\|v_k\| + M(\Delta\psi)_k.$$

Recall that $q_E \in [\frac{1}{2}, 1[$ (so $\frac{1-q_E}{q_E} \leq 1$) and that $\lim_{k\to+\infty} \tilde{V}_k = 0$. We obtain by induction that for all $k \geq \tilde{K}$

$$\sum_{p=k}^{+\infty} \|v_{p+1}\| \leq \frac{\lambda}{1-\lambda}\|v_k\| + \frac{Mc_E}{1-\lambda}\tilde{V}_k^{1-q_E}. \tag{3.47}$$

Or equivalently,

$$\Delta_k \leq \frac{\lambda}{1-\lambda}(\Delta_{k-1} - \Delta_k) + \frac{Mc_E}{1-\lambda}\tilde{V}_k^{1-q_E}. \tag{3.48}$$

Denoting $c_2 = (c_E(1-q_E))^{\frac{1-q_E}{q_E}}$, then by (3.37) and (3.38)

$$\begin{aligned}
\tilde{V}_k^{1-q_E} &\leq c_2 \|\nabla E(x_k, v_k)\|^{\frac{1-q_E}{q_E}} \\
&\leq c_2 \delta_3^{\frac{1-q_E}{q_E}} (\|v_k\| + \|v_{k+1}\|)^{\frac{1-q_E}{q_E}} \\
&\leq c_2 \delta_3^{\frac{1-q_E}{q_E}} (\Delta_{k-1} - \Delta_k + \Delta_k - \Delta_{k+1})^{\frac{1-q_E}{q_E}} \\
&= c_2 \delta_3^{\frac{1-q_E}{q_E}} (\Delta_{k-1} - \Delta_{k+1})^{\frac{1-q_E}{q_E}}.
\end{aligned}$$

Plugging this into (3.48), and using that $\Delta_k \to 0$ and $\frac{1-q_E}{q_E} \leq 1$, then there exists and integer $\tilde{K}_1 \geq \tilde{K}$ such that for all $k \geq \tilde{K}_1$

$$\Delta_k \leq \frac{\lambda}{1-\lambda}(\Delta_{k-1} - \Delta_k)^{\frac{1-q_E}{q_E}} + \frac{M_1}{1-\lambda}(\Delta_{k-1} - \Delta_{k+1})^{\frac{1-q_E}{q_E}},$$

where $M_1 = c_E c_2 \delta_3^{\frac{1-q_E}{q_E}} M$. Taking the power $\frac{q_E}{1-q_E} \geq 1$ on both sides and using the fact that $\Delta_{k+1} \leq \Delta_k$, we have for all $k \geq \tilde{K}_1$

$$\Delta_k^{\frac{q_E}{1-q_E}} \leq M_2(\Delta_{k-1} - \Delta_{k+1}), \tag{3.49}$$

where we set $M_2 = (1-\lambda)^{-\frac{q_E}{1-q_E}} \max(\lambda, M_1)^{\frac{q_E}{1-q_E}}$, We now distinguish two cases:

- $q \in [0, 1/2]$, hence $q_E = \frac{1}{2}$: (3.49) then becomes

$$\Delta_k \leq M_2(\Delta_{k-1} - \Delta_{k+1}),$$

with $M_2 = (1-\lambda)^{-1} \max(\lambda, M_1)$ and $M_1 = \frac{c_E^2}{2} \frac{\delta_3^2}{\delta}$. Using again that $\Delta_{k+1} \leq \Delta_k$, we obtain that for $k \geq \tilde{K}_1$

$$\Delta_k \leq \frac{M_2}{1+M_2} \Delta_{k-2},$$

which implies

$$\Delta_k \leq \left(\frac{M_2}{1+M_2}\right)^{\frac{k-\tilde{K}_1}{2}} \Delta_{\tilde{K}_1} = \mathcal{O}(\rho^k),$$

for $\rho \overset{\text{def}}{=} \left(\frac{M_2}{1+M_2}\right)^{\frac{1}{2}} \in]0, 1[$.

- $q \in]\frac{1}{2}, 1[$, hence $q_E = q$: we define the function $h : \mathbb{R}_+^* \to \mathbb{R}$ by $h(s) = s^{-\frac{q}{1-q}}$. Let $R > 1$. Assume first that $h(\Delta_k) \leq Rh(\Delta_{k-1})$. Then from (3.49), we get

$$\begin{aligned} 1 &\leq M_2(\Delta_{k-1} - \Delta_{k+1})h(\Delta_k) \\ &\leq RM_2(\Delta_{k-1} - \Delta_{k+1})h(\Delta_{k-1}) \\ &\leq RM_2 \int_{\Delta_{k+1}}^{\Delta_{k-1}} h(s) ds \\ &\leq RM_2 \frac{1-q}{1-2q} \left(\Delta_{k-1}^{\frac{1-2q}{1-q}} - \Delta_{k+1}^{\frac{1-2q}{1-q}}\right). \end{aligned}$$

Setting $\nu = \frac{2q-1}{1-q} > 0$ and $M_3 = \frac{\nu}{RM_2} > 0$, one obtains

$$0 < M_3 \leq \Delta_{k+1}^{-\nu} - \Delta_{k-1}^{-\nu}. \tag{3.50}$$

Now assume that $h(\Delta_k) > Rh(\Delta_{k-1})$. Since h is decreasing and $\Delta_{k+1} \leq \Delta_k$, then $h(\Delta_{k+1}) > Rh(\Delta_{k-1})$. Set $q = R^{\frac{2q-1}{q}} > 1$, we directly have that

$$\Delta_{k+1}^{-\nu} > q\Delta_{k-1}^{-\nu}.$$

Since $q - 1 > 0$ and $\Delta_k^{-\nu} \to +\infty$ as $k \to +\infty$, there exists $M_4 > 0$ and a large enough integer $\tilde{K}_2 \geq \tilde{K}_1$ such that for every $k \geq \tilde{K}_2$ that satisfies our assumption ($h(\Delta_k) > Rh(\Delta_{k-1})$), we have

$$0 < M_4 \leq \Delta_{k+1}^{-\nu} - \Delta_{k-1}^{-\nu}. \tag{3.51}$$

Taking $M_5 = \min(M_3, M_4)$, (3.50) and (3.51) show that for all $k \geq \tilde{K}_2$

$$0 < M_5 \leq \Delta_{k+1}^{-\nu} - \Delta_{k-1}^{-\nu}.$$

Summing both sides from $\tilde{K}_2$ up to $K - 1 \geq \tilde{K}_2$, we obtain

$$M_5(K - \tilde{K}_2) \leq \Delta_K^{-\nu} - \Delta_{\tilde{K}_2}^{-\nu} + \Delta_{K-1}^{-\nu} - \Delta_{\tilde{K}_2-1}^{-\nu} \leq 2(\Delta_K^{-\nu} - \Delta_{\tilde{K}_2-1}^{-\nu}).$$

Therefore

$$\Delta_K^{-\nu} \geq \Delta_{\tilde{K}_2-1}^{-\nu} + \frac{M_5}{2}(K - \tilde{K}_2). \tag{3.52}$$

Inverting, we get

$$\Delta_K \leq \left[\Delta_{\tilde{K}_2-1}^{-\nu} + \frac{M_5}{2}(K - (\tilde{K}_2 + 1))\right]^{-\frac{1}{\nu}} = \mathcal{O}(K^{-\frac{1}{\nu}}). \tag{3.53}$$

□

3.2.3 General coefficients

The discrete scheme (ISEHD-Disc) opens the question of whether we can consider α_k, β_k, s_k to be independent. Though this would omit the fact they arise from a discretization of the continuous-time dynamic (ISEHD), hence ignoring its physical interpretation, it will gives us a more flexible choice of these parameters while preserving the desired convergence behavior.

Theorem 3.10. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be satisfying (H_0) with ∇f being globally L-Lipschitz-continuous. Consider $(\alpha_k)_{k\in\mathbb{N}}, (\beta_k)_{k\in\mathbb{N}}, (s_k)_{k\in\mathbb{N}}$ to be three positive sequences, and the following algorithm with $x_0, x_1 \in \mathbb{R}^d$:*

$$\begin{cases} y_k &= x_k + \alpha_k(x_k - x_{k-1}) - \beta_k(\nabla f(x_k) - \nabla f(x_{k-1})), \\ x_{k+1} &= y_k - s_k \nabla f(x_k). \end{cases} \tag{3.54}$$

If there exists $\bar{s} > 0$ such that:

- $0 < \inf_{k\in\mathbb{N}} s_k \leq \sup_{k\in\mathbb{N}} s_k \leq \bar{s} < \frac{2}{L}$;
- $\sup_{k\in\mathbb{N}} \left(\frac{\alpha_k + \beta_k L}{s_k}\right) < \frac{1}{\bar{s}} - \frac{L}{2}$.

Then the following holds:

(i) $(\|\nabla f(x_k)\|)_{k\in\mathbb{N}} \in \ell^2(\mathbb{N})$, *and* $(\|x_{k+1} - x_k\|)_{k\in\mathbb{N}} \in \ell^2(\mathbb{N})$, *and thus*

$$\lim_{k\to+\infty} \|\nabla f(x_k)\| = 0.$$

(ii) Moreover, if $(x_k)_{k\in\mathbb{N}}$ is bounded and f is definable, then $(\|x_{k+1} - x_k\|)_{k\in\mathbb{N}} \in \ell^1(\mathbb{N})$ and x_k converges (as $k \to +\infty$) to a critical point of f.

(iii) Furthermore, if $\alpha_k \equiv \alpha, \beta_k \equiv \beta, s_k \equiv s$, then the previous conditions reduce to

$$\alpha + \beta L + \frac{sL}{2} < 1.$$

If, in addition, $\alpha \neq \frac{\beta}{\beta+s}$, and $\alpha > \beta L$, then for almost all $x_0, x_1 \in \mathbb{R}^d$, x_k converges (as $k \to +\infty$) to a critical point of f that is not a strict saddle. Consequently, if f satisfies the strict saddle property, for almost all $x_0, x_1 \in \mathbb{R}^d$, x_k converges (as $k \to +\infty$) to a local minimum of f.

Remark 3.11. If α_k, β_k, s_k are given as in (ISEHD-Disc), *i.e.* $\alpha_k = \frac{1}{1+\gamma_k h}, \beta_k = \beta h \alpha_k, s_k = h^2 \alpha_k$, then the requirements of Theorem 3.10 reduce to $\beta + \frac{h}{2} < \frac{c}{L}$ (recall that c is such that $c \leq \gamma_k$).

Proof. Adjusting equation (3.28) to this setting, *i.e.* not using the dependent explicit forms of α_k, β_k, s_k, we get an analogous proof to the one of Theorem 3.7. We omit the details for the sake of brevity. □

4 Inertial System with Implicit Hessian Damping

4.1 Continuous-time dynamics

We now turn to the second-order system with implicit Hessian damping as stated in (ISIHD), where we consider a constant geometric damping, *i.e.* $\beta(t) \equiv \beta > 0$. We will use the following equivalent reformulation of (ISIHD) proposed in [39]. We will say that x is a solution trajectory of (ISIHD) with initial conditions $x(0) = x_0, \dot{x}(0) = v_0$, if and only if, $x \in C^2(\mathbb{R}_+; \mathbb{R}^d)$ and there exists $y \in C^1(\mathbb{R}_+; \mathbb{R}^d)$ such that (x, y) satisfies:

$$\begin{cases} \dot{x}(t) + \frac{x(t)-y(t)}{\beta} & = 0, \\ \dot{y}(t) + \beta \nabla f(y(t)) + \left(\frac{1}{\beta} - \gamma(t)\right)(x(t) - y(t)) & = 0, \end{cases} \tag{4.1}$$

with initial conditions $x(0) = x_0, y(0) = y_0 \overset{\text{def}}{=} x_0 + \beta v_0$.

4.1.1 Global convergence of the trajectory

Our next main result is the following theorem, which is the implicit counterpart of Theorem 3.1.

Theorem 4.1. *Let* $0 < \beta < \frac{2c}{C^2}$, $f : \mathbb{R}^d \to \mathbb{R}$ *satisfying* (H_0), γ *is continuous and satisfies* (H_γ). *Consider* (ISIHD) *in this setting, then the following holds:*

(i) There exists a global solution trajectory $x : \mathbb{R}_+ \to \mathbb{R}^d$ *of* (ISIHD).

(ii) We have that $\dot{x} \in \mathrm{L}^2(\mathbb{R}_+; \mathbb{R}^d)$, *and* $\nabla f \circ (x + \beta \dot{x}) \in \mathrm{L}^2(\mathbb{R}_+; \mathbb{R}^d)$.

(iii) If we suppose that the solution trajectory x *is bounded over* $\mathbb{R}_+$, *then* $\nabla f \circ x \in \mathrm{L}^2(\mathbb{R}_+; \mathbb{R}^d)$,

$$\lim_{t \to +\infty} \|\nabla f(x(t))\| = \lim_{t \to +\infty} \|\dot{x}(t)\| = 0,$$

and $\lim_{t \to +\infty} f(x(t))$ *exists.*

(iv) In addition to (iii), if we also assume that f *is definable, then* $\dot{x} \in \mathrm{L}^1(\mathbb{R}_+; \mathbb{R}^d)$ *and* $x(t)$ *converges (as* $t \to +\infty$*) to a critical point of* f.

Proof. (i) We will start by showing the existence of a solution. Setting $Z = (x, y)$, (4.1) can be equivalently written as:

$$\dot{Z}(t) + \nabla \mathcal{G}(Z(t)) + \mathcal{D}(t, Z(t)) = 0, \quad Z(0) = (x_0, y_0), \tag{4.2}$$

where $\mathcal{G}(Z) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is the function defined by $\mathcal{G}(Z) = \beta f(y)$ and the time-dependent operator $\mathcal{D} : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ is given by:

$$\mathcal{D}(t, Z) = \left(\frac{x - y}{\beta}, \left(\frac{1}{\beta} - \gamma(t)\right)(x - y)\right).$$

Since the map $(t, Z) \mapsto \nabla \mathcal{G}(Z) + \mathcal{D}(t, Z)$ is continuous in the first variable and locally Lipschitz in the second (by (H_0) and the assumptions on γ),we get from Cauchy-Lipschitz theorem that there exists $T_{\max} > 0$ and a unique maximal solution of (3.2) denoted $Z \in C^1([0, T_{\max}[; \mathbb{R}^d \times \mathbb{R}^d)$. Consequently, there exists a unique maximal solution of (ISIHD) $x \in C^2([0, T_{\max}[; \mathbb{R}^d)$.

Let us consider the energy function $V : [0, T_{\max}[\to \mathbb{R}$ defined by

$$V(t) = f(x(t) + \beta \dot{x}(t)) + \frac{1}{2}\|\dot{x}(t)\|^2.$$

Proceeding as in the proof of Theorem 3.1, we prove it is indeed a Lyapunov function for (ISIHD). Denoting $\delta_1 \overset{\text{def}}{=} \min\left(\frac{c}{2}, \beta\left(1 - \frac{\beta C^2}{2c}\right)\right) > 0$, we have

$$V'(t) \le -\delta_1(\|\dot{x}(t)\|^2 + \|\nabla f(x(t) + \beta\dot{x}(t))\|^2). \tag{4.3}$$

We will now show that the maximal solution Z of (4.2) is actually global. For this, we argue by contradiction and assume that $T_{\max} < +\infty$. It is sufficient to prove that x and y have a limit as $t \to T_{\max}$, and local existence will contradict the maximality of $T_{\max}$. Integrating (4.3), we obtain $\dot{x} \in \mathrm{L}^2([0, T_{\max}[; \mathbb{R}^d)$ and $\nabla f \circ (x + \beta\dot{x}) \in \mathrm{L}^2([0, T_{\max}[; \mathbb{R}^d)$, which entails that $\dot{x} \in \mathrm{L}^1([0, T_{\max}[; \mathbb{R}^d)$ and $\nabla f \circ (x + \beta\dot{x}) \in \mathrm{L}^1([0, T_{\max}[; \mathbb{R}^d)$, and in turn $(x(t))_{t \in [0, T_{\max}[}$ satisfies the Cauchy property and $\lim_{t \to T_{\max}} x(t)$ exists. Besides, by the first equation of (4.1), we will have that $\lim_{t \to T_{\max}} y(t)$ will exist if both $\lim_{t \to T_{\max}} x(t)$ and $\lim_{t \to T_{\max}} \dot{x}(t)$ exist. So we just have to check the existence of the second limit. A sufficient condition would be to prove that $\ddot{x} \in \mathrm{L}^1([0, T_{\max}[; \mathbb{R}^d)$. By (ISIHD) this will hold if $\dot{x}, \nabla f \circ (x + \beta\dot{x})$ are in $\mathrm{L}^1([0, T_{\max}[; \mathbb{R}^d)$. But we have already shown these claims. Consequently, the solution Z of (4.2) is global, and thus the solution x of (ISIHD) is also global.

(ii) Integrating (4.3), using that V is well-defined and bounded from below, we get that $\dot{x} \in \mathrm{L}^2(\mathbb{R}_+; \mathbb{R}^d)$, and $\nabla f(x(t) + \beta\dot{x}(t)) \in \mathrm{L}^2(\mathbb{R}_+; \mathbb{R}^d)$.

(iii) By assumption, $\sup_{t>0} \|x(t)\| < +\infty$. Moreover, since $\dot{x} \in \mathrm{L}^2(\mathbb{R}_+; \mathbb{R}^d)$ and continuous, $\dot{x} \in \mathrm{L}^\infty(\mathbb{R}_+; \mathbb{R}^d)$ and then using that ∇f is locally Lipschitz, we have

$$\begin{aligned}
\int_0^{+\infty} \|\nabla f(x(t))\|^2 dt &\le 2\int_0^{+\infty} \|\nabla f(x(t) + \beta\dot{x}(t)) - \nabla f(x(t))\|^2 dt \\
&\quad + 2\int_0^{+\infty} \|\nabla f(x(t) + \beta\dot{x}(t))\|^2 dt \\
&\le 2\beta^2 L_0^2 \int_0^{+\infty} \|\dot{x}(t)\|^2 dt + 2\int_0^{+\infty} \|\nabla f(x(t) + \beta\dot{x}(t))\|^2 dt < +\infty,
\end{aligned}$$

where L_0 is the Lipschitz constant of ∇f on the centered ball of radius

$$\sup_{t>0} \|x(t)\| + \beta \sup_{t>0} \|\dot{x}(t)\| < +\infty.$$

Moreover, for every $t, s \ge 0$,

$$\|\nabla f(x(t)) - \nabla f(x(s))\| \le L_0 \sup_{\tau \ge 0} \|\dot{x}(\tau)\| \, |t - s|.$$

This combined with $\nabla f \circ x \in \mathrm{L}^2(\mathbb{R}_+; \mathbb{R}^d)$ yields

$$\lim_{t \to +\infty} \|\nabla f(x(t))\| = 0.$$

We also have that

$$\begin{aligned}
\sup_{t>0} \|\nabla f(x(t) + \beta\dot{x}(t))\| &\le \sup_{t>0}(\|\nabla f(x(t) + \beta\dot{x}(t)) - \nabla f(0)\|) + \|\nabla f(0)\| \\
&\le L_0 \sup_{t>0} \|x(t)\| + L_0\beta \sup_{t>0} \|\dot{x}(t)\| + \|\nabla f(0)\| < +\infty.
\end{aligned}$$

Therefore, in view of (ISIHD), we get that $\ddot{x} \in \mathrm{L}^\infty(\mathbb{R}_+;\mathbb{R}^d)$. This implies that

$$\|\dot{x}(t) - \dot{x}(s)\| \leq \sup_{\tau \geq 0} \|\ddot{x}(\tau)\| \, |t-s|.$$

Combining this with $\dot{x} \in \mathrm{L}^2(\mathbb{R}_+;\mathbb{R}^d)$ gives that $\lim_{t\to+\infty} \|\dot{x}(t)\| = 0$.
From (4.3), V is non-increasing, and since it is bounded from below, $V(t)$ has a limit, say $\tilde{L}$. Passing to the limit in the definition of $V(t)$, using that the velocity vanishes, gives $\lim_{t\to+\infty} f(x(t) + \beta\dot{x}(t)) = \tilde{L}$. On the other hand, we have

$$\begin{aligned} |f(x(t)+\beta\dot{x}(t)) - f(x(t))| &= \beta \left| \int_0^1 \langle \nabla f(x(t) + s\beta\dot{x}(t)), \dot{x}(t)\rangle \, ds \right| \\ &\leq \beta \left(\int_0^1 \|\nabla f(x(t) + s\beta\dot{x}(t))\| \, ds \right) \|\dot{x}(t)\| . \end{aligned}$$

Passing to the limit as $t \to +\infty$, the right hand side goes to 0 from the above limits on $\nabla f(x(t))$ and $\dot{x}(t)$. We deduce that $\lim_{t\to+\infty} f(x(t)) = \tilde{L}$.

(iv) As in the proof for (ISEHD), since $(x(t))_{t\geq 0}$ is bounded, then (3.4) holds. Besides, consider the function

$$E : (x,v,w) \in \mathbb{R}^{3d} \mapsto f(x+v) + \frac{1}{2}\|w\|^2 . \tag{4.4}$$

Since f is definable, so is E. In turn, E satisfies has the KŁ property. Let $\mathfrak{C}_1 = \mathfrak{C}(x(\cdot)) \times \{0_d\} \times \{0_d\}$. Since $E\big|_{\mathfrak{C}_1} = \tilde{L}, \nabla E\big|_{\mathfrak{C}_1} = 0, \exists r, \eta > 0, \exists \psi \in \kappa(0,\eta)$ such that for every $(x,v,w) \in \mathbb{R}^{3d}$ such that $x \in \mathfrak{C}(x(\cdot)) + B_r, v \in B_r, w \in B_r$ and $0 < E(x,v,w) - \tilde{L} < \eta$, we have

$$\psi'(E(x,v,w) - \tilde{L})\|\nabla E(x,v,w)\| \geq 1 \tag{4.5}$$

By definition, we have $V(t) = E(x(t), \beta\dot{x}(t), \dot{x}(t)$. We also define $\tilde{V}(t) = V(t) - \tilde{L}$. By the properties of V above, we have $\lim_{t\to+\infty} \tilde{V}(t) = 0$ and $\tilde{V}$ is a non-increasing function. Thus $\tilde{V}(t) \geq 0$ for every $t > 0$. Without loss of generality, we may assume that $\tilde{V}(t) > 0$ for every $t > 0$ (since otherwise $\tilde{V}(t)$ is eventually zero entailing that $\dot{x}(t)$ is eventually zero in view of (4.3), meaning that $x(\cdot)$ has finite length).
Define the constants $\delta_2 = 2, \delta_3 = \frac{\delta_1}{\sqrt{2}}$. In view of the convergence claims on $\dot{x}$ and $\tilde{V}$ above, there exists $T > 0$, such that for any $t > T$

$$\begin{cases} x(t) \in \mathfrak{C}(x(\cdot)) + B_r, \\ 0 < \tilde{V}(t) < \eta, \\ \max(\beta, 1)\|\dot{x}(t)\| < r, \\ \frac{1}{\delta_3}\psi(\tilde{V}(t)) < \frac{r}{2\sqrt{2}}. \end{cases} \tag{4.6}$$

The rest of the proof is analogous to the one of Theorem 3.1. Since

$$\|\nabla E(x(t), \beta\dot{x}(t), \dot{x})\|^2 \leq \delta_2(\|\dot{x}(t)\|^2 + \|\nabla f(x(t) + \beta\dot{x}(t))\|^2), \tag{4.7}$$

and

$$\psi'(\tilde{V}(t))\|\nabla E(x(t), \beta\dot{x}(t), \dot{x})\| \geq 1, \quad \forall t \geq T. \tag{4.8}$$

We can lower bound the term $-\frac{d}{dt}\psi(\tilde{V}(t))$ for $t \geq T$ (as in (3.12)) and conclude that $\dot{x} \in \mathrm{L}^1(\mathbb{R}_+;\mathbb{R}^d)$, and that this implies that $x(t)$ has finite length and thus has a limit as $t \to +\infty$. This limit is necessarily a critical point of f since $\lim_{t\to+\infty} \|\nabla f(x(t))\| = 0$.

□

4.1.2 Trap avoidance

We now show that (ISIHD) provably avoids strict saddle points, hence implying convergence to a local minimum if the objective function is Morse.

Theorem 4.2. *Let $c > 0$, $0 < \beta < \frac{2}{c}$ and $\gamma \equiv c$. Assume that $f : \mathbb{R}^d \to \mathbb{R}$ satisfies (H_0) and is a Morse function. Consider (ISIHD) in this setting. If the solution trajectory x is bounded over $\mathbb{R}_+$, then the conclusions of Theorem 4.1 hold. If, moreover, $\beta \neq \frac{1}{c}$, then for almost all $x_0, v_0 \in \mathbb{R}^d$ initial conditions, $x(t)$ converges (as $t \to +\infty$) to a local minimum of f.*

Proof. Since Morse functions are C^2 and satisfy the KŁ inequality, and x is assumed bounded, then all the claims of Theorem 4.1 hold.

As in the proof of Theorem 3.3, we will use again the global stable manifold theorem to prove the last point. Since, $\gamma(t) = c$ for all t, introducing the velocity variable $v = \dot{x}$, we have the equivalent phase-space formulation of (ISIHD)

$$\begin{cases} \dot{x}(t) &= v(t), \\ \dot{v}(t) &= -cv(t) - \nabla f(x(t) + \beta v(t)), \end{cases} \tag{4.9}$$

with initial conditions $x(0) = x_0, v(0) = v_0$. Let us consider $F : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ defined by

$$F(x, y) = (v, -cv - \nabla f(x + \beta v)) .$$

Defining $z(t) = (x(t), v(t))$ and $z_0 = (x_0, v_0) \in \mathbb{R}^{2d}$, then (4.9) is equivalent to

$$\begin{cases} \dot{z}(t) &= F(z(t)), \\ z(0) &= z_0. \end{cases} \tag{4.10}$$

We know from above that under our conditions, the solution trajectory $z(t)$ converges (as $t \to +\infty$) to an equilibrium point of F, and the set of equilibria is $\{(\hat{x}, 0) : \hat{x} \in \mathrm{crit}(f)\}$. Following the same ideas as in the proof of Theorem 3.3, first, we will prove that each equilibrium point of F is hyperbolic. We first compute the Jacobian

$$J_F(x, y) = \begin{pmatrix} 0_{d\times d} & I_d \\ -\nabla^2 f(x + \beta v) & -cI_d - \beta\nabla^2 f(x + \beta v) \end{pmatrix} .$$

Let $\hat{z} = (\hat{x}, 0)$, where $\hat{x} \in \mathrm{crit}(f)$. Then the eigenvalues of $J_F(\hat{z})$ are characterized by the solutions on $\lambda \in \mathbb{C}$ of

$$\det\left(\begin{pmatrix} -\lambda I_d & I_d \\ -\nabla^2 f(\hat{x}) & -(\lambda + c)I_d - \beta\nabla^2 f(\hat{x}) \end{pmatrix}\right) = 0. \tag{4.11}$$

By Lemma 2.1, (4.11) is equivalent to

$$\det((1 + \lambda\beta)\nabla^2 f(\hat{x}) + (\lambda^2 + \lambda c)I_d) = 0. \tag{4.12}$$

This is the exact same equation as (3.17). Thus the rest of the analysis goes as in the proof of Theorem 3.3. □

4.1.3 Convergence rate

We now give asymptotic convergence rates on the objective and trajectory.

Theorem 4.3. *Consider the setting of Theorem 4.1 with f being also definable. Recall the function E from (4.4), which is also definable, and denote ψ its desingularizing function and Ψ any primitive of $-\psi'^2$. Then, $x(t)$ converges (as $t \to +\infty$) to $x_\infty \in \mathrm{crit}(f)$. Denote $\tilde{V}(t) \stackrel{\text{def}}{=} E(x(t), \beta\dot{x}(t), \dot{x}(t)) - f(x_\infty)$. Then, the following rates of convergence hold:*

- *If $\lim_{t\to 0} \Psi(t) \in \mathbb{R}$, we have $E(x(t), \dot{x}(t), \beta\nabla f(x(t)))$ converges to $f(x_\infty)$ in finite time.*
- *If $\lim_{t\to 0} \Psi(t) = +\infty$, there exists some $t_1 \geq 0$ such that*

$$\tilde{V}(t) = \mathcal{O}(\Psi^{-1}(t - t_1)) \tag{4.13}$$

Moreover,

$$\|x(t) - x_\infty\| = \mathcal{O}(\psi \circ \Psi^{-1}(t - t_1)) \tag{4.14}$$

Proof. Analogous to Theorem 3.4. □

When f has the Łojasiewicz property, we get the following corollary of Theorem 4.3.

Corollary 4.4. *Consider the setting of Theorem 4.3 where now f satisfies the Łojasiewicz inequality with desingularizing function $\psi_f(s) = c_f s^{1-q}$, $q \in [0, 1[$, $c_f > 0$. Then there exists some $t_1 > 0$ such that:*

- *If $q \in [0, \frac{1}{2}]$, then*

$$\tilde{V}(t) = \mathcal{O}(\exp(-(t - t_1))) \quad \textit{and} \quad \|x(t) - x_\infty\| = \mathcal{O}\left(\exp\left(\frac{t - t_1}{2}\right)\right) \tag{4.15}$$

- *If $q \in]\frac{1}{2}, 1[$, then*

$$\tilde{V}(t) = \mathcal{O}((t - t_1)^{\frac{-1}{2q-1}}) \quad \textit{and} \quad \|x(t) - x_\infty\| = \mathcal{O}((t - t_1)^{-\frac{1-q}{2q-1}}) \tag{4.16}$$

Proof. Analogous to Corollary 3.6. □

4.2 Algorithmic scheme

In this section, we will study the properties of an algorithmic scheme derived from the following explicit discretization discretization of (ISIHD) with step-size $h > 0$ and for $k \geq 1$:

$$\frac{x_{k+1} - 2x_k + x_{k-1}}{h^2} + \gamma(kh)\frac{x_{k+1} - x_k}{h} + \nabla f\left(x_k + \beta\frac{x_k - x_{k-1}}{h}\right) = 0. \tag{4.17}$$

This is equivalently written as

$$\begin{cases} y_k &= x_k + \alpha_k(x_k - x_{k-1}), \\ x_{k+1} &= y_k - s_k\nabla f(x_k + \beta'(x_k - x_{k-1})), \end{cases} \tag{ISIHD-Disc}$$

with initial conditions $x_0, x_1 \in \mathbb{R}^d$, where $\alpha_k \stackrel{\text{def}}{=} \frac{1}{1+\gamma_k h}$, $s_k \stackrel{\text{def}}{=} h^2\alpha_k$ and $\beta' \stackrel{\text{def}}{=} \frac{\beta}{h}$.

4.2.1 Global convergence and trap avoidance

We have the following result which characterizes the asymptotic behavior of algorithm (ISIHD-Disc), which shows that the latter enjoys the same guarantees as (ISEHD-Disc) given in Theorem 3.7. We will again require that ∇f is globally Lipschitz-continuous.

Theorem 4.5. *Let $f : \mathbb{R}^d \to \mathbb{R}$ satisfying (H_0) with ∇f being globally L-Lipschitz-continuous. Consider algorithm (ISIHD-Disc) with $h > 0$, $\beta \geq 0$ and $c \leq \gamma_k \leq C$ for some $c, C > 0$ and for every $k \in \mathbb{N}$. Then the following holds:*

(i) If $\beta + \frac{h}{2} < \frac{c}{L}$, then $(\|\nabla f(x_k)\|)_{k\in\mathbb{N}} \in \ell^2(\mathbb{N})$, and $(\|x_{k+1} - x_k\|)_{k\in\mathbb{N}} \in \ell^2(\mathbb{N})$, in particular

$$\lim_{k\to+\infty} \|\nabla f(x_k)\| = 0.$$

(ii) Moreover, if $(x_k)_{k\in\mathbb{N}}$ is bounded and f is definable, then $(\|x_{k+1} - x_k\|)_{k\in\mathbb{N}} \in \ell^1(\mathbb{N})$ and x_k converges (as $k \to +\infty$) to a critical point of f.

(iii) Furthermore, if $\gamma_k \equiv c > 0$, $0 < \beta < \frac{c}{L}, \beta \neq \frac{1}{c}$, and $h < \min\left(2\left(\frac{c}{L} - \beta\right), \frac{1}{L\beta}\right)$, then for almost all $x_0, x_1 \in \mathbb{R}^d$, x_k converges (as $k \to +\infty$) to a critical point of f that is not a strict saddle. Consequently, if f satisfies the strict saddle property, for almost all $x_0, x_1 \in \mathbb{R}^d$, x_k converges (as $k \to +\infty$) to a local minimum of f.

Proof. (i) Let $v_k \stackrel{\text{def}}{=} x_k - x_{k-1}$, $\bar{\alpha} \stackrel{\text{def}}{=} \frac{1}{1+ch}$, $\underline{\alpha} \stackrel{\text{def}}{=} \frac{1}{1+Ch}$, $\bar{s} = h^2\bar{\alpha}, \underline{s} = h^2\underline{\alpha}$, so $\underline{\alpha} \leq \alpha_k \leq \bar{\alpha}$ and $\underline{s} \leq s_k \leq \bar{s}$ for every $k \in \mathbb{N}$. Proceeding as in the proof of Theorem 3.7, we have by definition that for $k \in \mathbb{N}^*$

$$x_{k+1} = \operatorname*{argmin}_{x\in\mathbb{R}^d} \frac{1}{2}\|x - (y_k - s_k\nabla f(x_k + \beta' v_k))\|^2, \tag{4.18}$$

and $1-$strong convexity of $x \mapsto \frac{1}{2}\|x - (y_k - s_k\nabla f(x_k + \beta' v_k))\|^2$ then gives

$$\frac{1}{2}\|x_{k+1} - (y_k - s_k\nabla f(x_k + \beta' v_k))\|^2 \leq \frac{1}{2}\|x_k - (y_k - s_k\nabla f(x_k + \beta' v_k))\|^2 - \frac{1}{2}\|x_{k+1} - x_k\|^2. \tag{4.19}$$

Expanding and rearranging, we obtain

$$\langle \nabla f(x_k + \beta' v_k), v_{k+1}\rangle \leq -\frac{\|v_{k+1}\|^2}{s_k} + \frac{1}{h^2}\langle v_k, v_{k+1}\rangle. \tag{4.20}$$

Combining this with the descent lemma of L-smooth functions applied to f, we arrive at

$$\begin{aligned} f(x_{k+1}) &\leq f(x_k) + \langle \nabla f(x_k), v_{k+1}\rangle + \frac{L}{2}\|v_{k+1}\|^2 \\ &= f(x_k) + \langle \nabla f(x_k) - \nabla f(x_k + \beta' v_k), v_{k+1}\rangle + \langle \nabla f(x_k + \beta' v_k), v_{k+1}\rangle + \frac{L}{2}\|v_{k+1}\|^2 \\ &\leq f(x_k) + \left(\beta' L + \frac{1}{h^2}\right)\|v_k\|\|v_{k+1}\| - \left(\frac{1}{\bar{s}} - \frac{L}{2}\right)\|v_{k+1}\|^2. \end{aligned}$$

Where we have used that the gradient of f is $L-$Lipschitz and Cauchy-Schwarz inequality in the last bound. Denote $\tilde{\alpha} = \beta' L + \frac{1}{h^2}$. We can check that since $h < \frac{2c}{L}$, then $0 < \bar{s} < \frac{2}{L}$ and the last term of the inequality is negative. Using Young's inequality we have that for $\varepsilon > 0$:

$$f(x_{k+1}) \leq f(x_k) + \frac{\tilde{\alpha}^2}{2\varepsilon}\|v_k\|^2 + \varepsilon\frac{\|v_{k+1}\|^2}{2} - \left(\frac{1}{\bar{s}} - \frac{L}{2}\right)\|v_{k+1}\|^2.$$

Or equivalently,

$$f(x_{k+1})+\frac{\tilde{\alpha}^2}{2\varepsilon}\|v_{k+1}\|^2\leq f(x_k)+\frac{\tilde{\alpha}^2}{2\varepsilon}\|v_k\|^2+\left[\frac{\varepsilon}{2}+\frac{\tilde{\alpha}^2}{2\varepsilon}-\left(\frac{1}{\bar{s}}-\frac{L}{2}\right)\right]\|v_{k+1}\|^2.$$

In order to make the last term negative, we impose

$$\frac{\varepsilon}{2}+\frac{\tilde{\alpha}^2}{2\varepsilon}<\frac{1}{\bar{s}}-\frac{L}{2}.$$

Minimizing for ε at the left-hand side we obtain $\varepsilon=\tilde{\alpha}$ and the condition to satisfy is

$$\bar{s}<\frac{2}{2\tilde{\alpha}+L}. \tag{4.21}$$

Recalling the definitions of $\bar{s},\tilde{\alpha},\beta'$, this is equivalent to

$$\frac{h^2}{1+ch}<\frac{2}{2\left(L\frac{\beta}{h}+\frac{1}{h^2}\right)+L}\iff 2L\beta h+2+Lh^2<2+2ch.$$

Simplifying, this reads

$$\beta+\frac{h}{2}<\frac{c}{L},$$

which is precisely what we have assumed. Let $\delta=\left(\frac{1}{\bar{s}}-\frac{L}{2}\right)-\tilde{\alpha}>0$, then

$$f(x_{k+1})+\frac{\tilde{\alpha}}{2}\|v_{k+1}\|^2\leq f(x_k)+\frac{\tilde{\alpha}}{2}\|v_k\|^2-\delta\|v_{k+1}\|^2. \tag{4.22}$$

Toward our Lyapunov analysis, define now $V_k=f(x_k)+\frac{\tilde{\alpha}}{2}\|v_k\|^2$ for $k\in\mathbb{N}^*$. In view of (4.22), V_k obeys

$$V_{k+1}\leq V_k-\delta\|v_{k+1}\|^2. \tag{4.23}$$

and thus V_k is non-increasing. Since it is also bounded from below, V_k converges to a limit, say $\tilde{L}$. Summing (4.22) over $k\in\mathbb{N}^*$, we get that $(\|v_{k+1}\|)_{k\in\mathbb{N}}\in\ell^2(\mathbb{N})$, hence $\lim_{k\to+\infty}\|v_k\|=0$. Besides, since $\bar{\alpha}<1$

$$\begin{aligned}\|\nabla f(x_k+\beta' v_k)\|=\frac{1}{s_k}\|x_{k+1}-y_k\|&\leq\frac{1}{\underline{s}}(\|x_{k+1}-x_k\|+\|x_k-y_k\|)\\&\leq\frac{1}{\underline{s}}(\|v_{k+1}\|+\bar{\alpha}\|v_k\|)\\&\leq\frac{1}{\underline{s}}(\|v_{k+1}\|+\|v_k\|),\end{aligned}$$

which implies

$$\|\nabla f(x_k+\beta' v_k)\|^2\leq\delta_2(\|v_{k+1}\|^2+\|v_k\|^2),$$

where $\delta_2=\frac{2}{\underline{s}^2}$. Consequently $(\|\nabla f(x_k+\beta' v_k)\|)_{k\in\mathbb{N}}\in\ell^2(\mathbb{N})$, and

$$\begin{aligned}\|\nabla f(x_k)\|^2&=2(\|\nabla f(x_k)-\nabla f(x_k+\beta' v_k)\|^2+\|\nabla f(x_k+\beta' v_k)\|^2)\\&\leq 2(L^2\beta'^2\|v_k\|^2+\|\nabla f(x_k+\beta' v_k)\|^2).\end{aligned}$$

Thus, $(\|\nabla f(x_k)\|)_{k\in\mathbb{N}}\in\ell^2(\mathbb{N})$, hence $\lim_{k\to+\infty}\|\nabla f(x_k)\|=0$.

(ii) When $(x_k)_{k\in\mathbb{N}}$ is bounded and f is definable, we proceed analogously as in the proof of Theorem 3.7 to conclude that $(\|v_k\|)_{k\in\mathbb{N}} \in \ell^1(\mathbb{N}^*)$, so $(x_k)_{k\in\mathbb{N}}$ is a Cauchy sequence which implies that it has a limit (as $k \to +\infty$) denoted x_∞, which is a critical point of f since $\lim_{k\to+\infty} \|\nabla f(x_k)\| = 0$.

(iii) When $\gamma_k \equiv c$, we let $\alpha_k \equiv \alpha = \frac{1}{1+ch}$, $s_k \equiv s = h^2\alpha$. Let $z_k = (x_k, x_{k-1})$, and $g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ defined by

$$g : (x_+, x_-) \mapsto [(1+\alpha)x_+ - \alpha x_- - s\nabla f(x_+ + \beta'(x_+ - x_-)), x_+].$$

(ISIHD-Disc) is then equivalent to

$$z_{k+1} = g(z_k). \tag{4.24}$$

To conclude, we will again use [66, Corollary 1], similarly to what we did in the proof of of Theorem 3.7, by checking that:

(a) $\det(J_g(x_+, x_-)) \neq 0$ for every $x_+, x_- \in \mathbb{R}^d$.

(b) $\hat{\mathcal{X}} \subset \mathcal{A}_g^\star$, where $\mathcal{A}_g^\star \overset{\text{def}}{=} \{(x,x) \in \mathbb{R}^{2d} : x \in \mathrm{crit}(f), \max_i |\lambda_i(J_g(x,x))| > 1\}$ and $\hat{\mathcal{X}} \overset{\text{def}}{=} \{(x,x) \in \mathbb{R}^{2d} : x \in \mathcal{X}^\star\}$, with $\mathcal{X}^\star$ the set of strict saddle points of f.

The Jacobian $J_g(x_+, x_-)$ reads

$$\begin{pmatrix} (1+\alpha)I_d - s(1+\beta')\nabla^2 f(x_+ + \beta'(x_+ - x_-)) & -\alpha I_d + s\beta'\nabla^2 f(x_+ + \beta'(x_+ - x_-)) \\ I_d & 0_{d\times d} \end{pmatrix}. \tag{4.25}$$

This is a block matrix, where the bottom-left matrix commutes with the upper-left matrix (since is the identity matrix), then by Lemma 2.1 $\det(J_g(x_+, x_-)) = \det(\alpha I_d - \beta' s\nabla^2 f(x_+ + \beta'(x_+ - x_-)))$. Since the eigenvalues of $\nabla^2 f(x_+ + \beta'(x_+ - x_-))$ are contained in $[-L, L]$. It is then sufficient that $\alpha > \beta' L s$ to have that $\eta - \frac{\alpha}{\beta' s} \neq 0$ for every eigenvalue $\eta \neq 0$ of $\nabla^2 f(x_+ + \beta'(x_+ - x_-))$. This means that under $\alpha > \beta' L s$, condition (a) is in force. Requiring $\alpha > \beta' L s$ is equivalent to $h < \frac{1}{L\beta}$, and since we already need $h < 2\left(\frac{c}{L} - \beta\right)$, we just ask h to be less than the minimum of the two quantities.

Let us check condition (b). Let x be a strict saddle point of f, *i.e.* $x \in \mathrm{crit}(f)$ and $\lambda_{\min}(\nabla^2 f(x)) = -\eta < 0$. To characterize the eigenvalues of $J_g(x,x)$ we could use Lemma 2.1 as before, however, we will present an equivalent argument. Let $\eta_i \in \mathbb{R}$, $i = 1, \ldots, d$, be the eigenvalues of $\nabla^2 f(x)$. By symmetry of the Hessian, it is easy to see that the $2d$ eigenvalues of $J_g(x,x)$ coincide with the eigenvalues of the 2×2 matrices

$$\begin{pmatrix} (1+\alpha) - s(1+\beta')\eta_i & -\alpha + s\beta'\eta_i \\ 1 & 0 \end{pmatrix}.$$

These eigenvalues are therefore the (complex) roots of

$$\lambda^2 - \lambda\left((1+\alpha) - s(1+\beta')\eta_i\right) + \alpha - s\beta'\eta_i = 0. \tag{4.26}$$

If $\lambda = \frac{\beta'}{\beta'+1}$, then (4.26) becomes

$$\left(\frac{\beta'}{\beta'+1}\right)^2 - \left(\frac{\beta'}{\beta'+1}\right)(1+\alpha) + \alpha = 0.$$

This implies that $\alpha = \frac{\beta'}{\beta'+1}$, or equivalently $\frac{1}{1+ch} = \frac{\beta}{\beta+h}$. But this contradicts our assumption that $\beta c \neq 1$, and thus this case cannot occur. Let us now solve (4.26) for $\eta_i = -\eta$. Its discriminant is

$$\Delta_\lambda = \alpha^2 + 2\alpha(\eta s(1+\beta') - 1) + (\eta s(1+\beta') + 1)^2 - 4\eta s\beta',$$

which can be seen as a quadratic equation in α whose discriminant $\Delta_\alpha = -16\eta s$. Since $\Delta_\alpha < 0$ (recall that $\eta, s > 0$). Therefore the quadratic equation on α does not have real roots implying that $\Delta_\lambda > 0$. We can then write the solutions of (4.26),

$$\lambda = \frac{((1+\alpha) + \eta s(1+\beta')) \pm \sqrt{\Delta_\lambda}}{2}.$$

Let us examine the largest solution (the one with the plus sign) and show that actually $\lambda > 1$. Simple algebra shows that this is equivalent to verifying that

$$\Delta_\lambda > (2 - (1+\alpha) - \eta s(1+\beta'))^2.$$

or, equivalently,

$$(1-\alpha)^2 + \eta^2 s^2 (1+\beta')^2 - 2(1-\alpha)\eta s(1+\beta') < \Delta_\lambda \\ = \left((1+\alpha)^2 + \eta(s-\beta')\right)^2 - 4(\alpha - \eta\beta').$$

Simple algebra again shows that this inequality is equivalent to $0 < 4\eta s$, which is always true as $\eta > 0$. We have thus shown that $\hat{\mathcal{X}} \subset \mathcal{A}_g^\star$.

Overall, we have checked the two conditions (a)-(b) above. Therefore the invariant set $\{z_1 \in \mathbb{R}^{2d} : \lim_{k\to+\infty} g^k(z_1) \in \hat{\mathcal{X}}\}$ has Lebesgue measure zero. This means that the set of initializations $x_0, x_1 \in \mathbb{R}^d$ for which x_k converges to a strict saddle point of f has Lebesgue measure zero.

□

4.2.2 Convergence rate

The asymptotic convergence rate of algorithm (ISIHD-Disc) for Łojasiewicz functions is given in the following theorem. This shows that (ISIHD-Disc) enjoys the same asymptotic convergence rates as (ISEHD-Disc).

Theorem 4.6. *Consider the setting of Theorem 4.5, where f also satisfies the Łojasiewicz property with exponent $q \in [0, 1[$. Then $x_k \to x_\infty \in \mathrm{crit}(f)$ as $k \to +\infty$ at the rates:*

- *If $q \in [0, \frac{1}{2}]$ then there exists $\rho \in]0, 1[$ such that*

$$\|x_k - x_\infty\| = \mathcal{O}(\rho^k). \tag{4.27}$$

- *If $q \in]\frac{1}{2}, 1[$ then*

$$\|x_k - x_\infty\| = \mathcal{O}\left(k^{-\frac{1-q}{2q-1}}\right). \tag{4.28}$$

Proof. Since the Lyapunov analysis of (ISIHD-Disc) is analogous to that of (ISEHD-Disc) (though the Lyapunov functions are different), the proof of this theorem is similar to the one of Theorem 3.9. □

4.2.3 General coefficients

As discussed for the explicit case, the discrete scheme (ISIHD-Disc) rises from a discretization of the ODE (ISIHD). However, the parameters α_k, s_k are linked to each other. We now consider (ISIHD-Disc) where α_k, s_k are independent. Though this would hide somehow the physical interpretation of these parameters, it allows for some flexibility in their choice while preserving the convergence behavior.

Theorem 4.7. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be satisfying* (H_0) *with ∇f being globally L-Lipschitz-continuous. Consider $(\alpha_k)_{k\in\mathbb{N}}, (\beta_k)_{k\in\mathbb{N}}, (s_k)_{k\in\mathbb{N}}$ to be three positive sequences, and the following algorithm with $x_0, x_1 \in \mathbb{R}^d$:*

$$\begin{cases} y_k &= x_k + \alpha_k(x_k - x_{k-1}), \\ x_{k+1} &= y_k - s_k \nabla f(x_k + \beta_k(x_k - x_{k-1})). \end{cases} \tag{4.29}$$

If there exists $\bar{s} > 0$ such that:

- $0 < \inf_{k\in\mathbb{N}} s_k \le \sup_{k\in\mathbb{N}} s_k \le \bar{s} < \frac{2}{L}$;
- $\sup_{k\in\mathbb{N}} \left(\beta_k L + \frac{\alpha_k}{s_k}\right) < \frac{1}{\bar{s}} - \frac{L}{2}$.

Then the following holds:

(i) $(\|\nabla f(x_k)\|)_{k\in\mathbb{N}} \in \ell^2(\mathbb{N})$, *and* $(\|x_{k+1} - x_k\|)_{k\in\mathbb{N}} \in \ell^2(\mathbb{N})$, *hence*

$$\lim_{k\to+\infty} \|\nabla f(x_k)\| = 0.$$

(ii) Moreover, if $(x_k)_{k\in\mathbb{N}}$ is bounded and f is definable, then $(\|x_{k+1} - x_k\|)_{k\in\mathbb{N}} \in \ell^1(\mathbb{N})$ and x_k converges (as $k \to +\infty$) to a critical point of f.

(iii) Furthermore, if $\alpha_k \equiv \alpha, \beta_k \equiv \beta, s_k \equiv s$, then the previous conditions reduce to

$$\alpha + sL\left(\beta + \frac{1}{2}\right) < 1.$$

If, in addition, $\alpha \neq \frac{\beta}{\beta+1}$, $\alpha > \beta L s$, then for almost all $x_0, x_1 \in \mathbb{R}^d$, x_k converges (as $k \to +\infty$) to a critical point of f that is not a strict saddle. Consequently, if f satisfies the strict saddle property, for almost all $x_0, x_1 \in \mathbb{R}^d$, x_k converges (as $k \to +\infty$) to a local minimum of f.

Proof. Adjusting equation (4.20) to our setting (*i.e.* not using the dependency of α_k, s_k) we get an analogous proof to the one of Theorem 4.5. We omit the details. □

5 Numerical experiments

Before describing the numerical experiments, let us start with a few observations on the computational complexity and memory storage requirement of (ISEHD-Disc) and (ISIHD-Disc). The number of gradient access per iteration is the same for Gradient Descent (GD), discrete HBF, (ISEHD-Disc) and (ISIHD-Disc) is the same (one per iteration). However, the faster convergence (in practice) of inertial methods comes at the cost of storing previous information. For the memory storage requirement per iteration, GD stores only the previous iterate, the discrete HBF and (ISIHD-Disc) store the two previous iterates, while (ISEHD-Disc) additionally stores the previous gradient iterate as well. This has to be kept in mind when comparing these algorithms especially for in very high dimensional settings.

We will illustrate our findings with two numerical experiments. The first one is the optimization of the Rosenbrock function in $\mathbb{R}^2$, while the second one is on image deblurring. We will apply the proposed discrete schemes (ISEHD-Disc) and (ISIHD-Disc) and compare them with gradient descent and the (discrete) HBF. We will call $\|\nabla f(x_k)\|$ the residual.

5.1 Rosenbrock function

We will minimize the classical Rosenbrock function, *i.e.*,

$$f : (x, y) \in \mathbb{R}^2 \mapsto (1 - x)^2 + 100(y - x^2)^2,$$

with global minimum at $(x^\star, y^\star) = (1, 1)$.

We notice that its global minimum is the only critical point. Therefore this function is Morse and thus satisfies the Łojasiewicz inequality with exponent $\frac{1}{2}$ (see Remark 2.13). Consider (ISEHD) and (ISIHD) with the Rosenbrock function as the objective, $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying (H_γ) (*i.e.* $0 < c \leq \gamma(t) \leq C < +\infty$), and $0 < \beta < \frac{2c}{C^2}$. By Theorems 3.1-3.9 for (ISEHD), and Theorems 4.1-4.6 for (ISIHD), we get that the solution trajectories of these dynamics will converge to the global minimum eventually at a linear rate. Due to the low dimensionality of this problem, we could use an ODE solver to show numerically these results. However, we will just the iterates generated by our proposed algorithmic schemes (ISEHD-Disc) and (ISIHD-Disc). Although the gradient of the objective is not globally Lipschitz continuous, our proposed algorithmic schemes worked very well for h small enough. This suggests that we may relax this hypothesis in future work, as proposed in [67, 68] for GD.

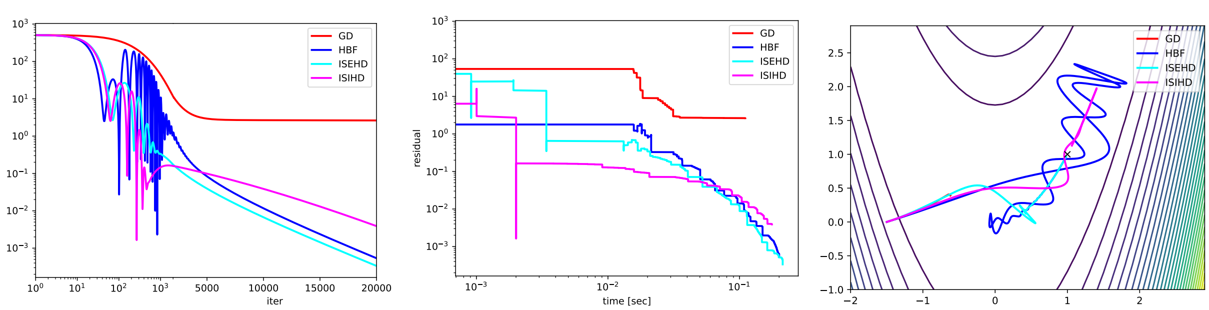
We applied (ISEHD-Disc) and (ISIHD-Disc) with $\beta \in \{0.02, 0.04\}$, $\gamma(t) \equiv \gamma_0 = 3$, $h = 10^{-3}$ and initial conditions $x_0 = (-1.5, 0), x_1 = x_0$. We compared our algorithms with GD and HBF (with the same initial conditions) after $2 * 10^4$ iterations:

$$x_{k+1} = x_k - \frac{h^2}{1 + \gamma_0 h} \nabla f(x_k), \tag{GD}$$

and

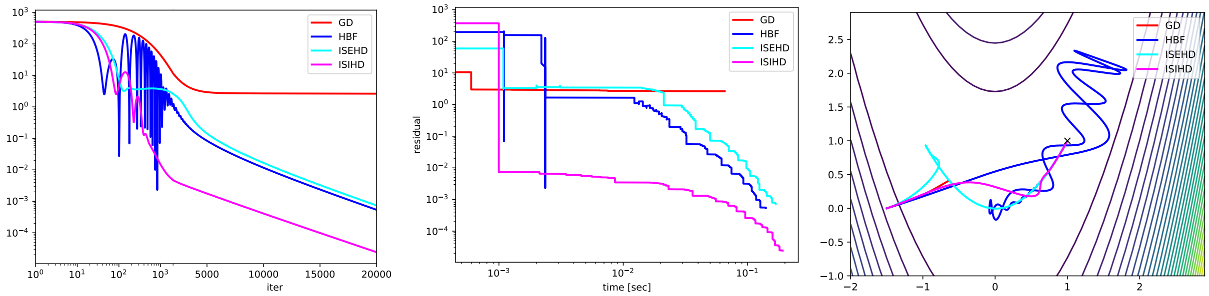
$$\begin{cases} y_k & = x_k + \frac{1}{1+\gamma_0 h}(x_k - x_{k-1}), \\ x_{k+1} & = y_k - \frac{h^2}{1+\gamma_0 h} \nabla f(x_k). \end{cases} \tag{HBF}$$

The behavior of all algorithms is depicted in Figures 1 and 2.



(a) Iteration vs Residual. (b) Time vs Residual. (c) Trajectory.

Figure 1: Results on the Rosenbrock function with $\beta = 0.02$.



(a) Iteration vs Residual. (b) Time vs Residual. (c) Trajectory.

Figure 2: Results on the Rosenbrock function with $\beta = 0.04$.

We can notice that the iterates generated by (ISEHD-Disc) and (ISIHD-Disc) oscillate much less towards the minimum than (HBF), and this damping effect is more notorious as β gets larger. In the case $\beta = 0.02$, we observe there are still some oscillations, which benefit the dynamic generated by (ISEHD-Disc) more than the one generated by (ISIHD-Disc). However, we have the opposite effect in the case $\beta = 0.04$, where the oscillations are more damped. These three methods ((ISEHD-Disc), (ISIHD-Disc), (HBF)) share a similar asymptotic convergence rate, which is linear as predicted (recall f is Łojasiewicz with exponent $1/2$), and they are significantly faster than (GD).

5.2 Image Deblurring

In the task of image deblurring, we are given a blurry and noisy (gray-scale) image $b \in \mathbb{R}^{n_x \times n_y}$ of size $n_x \times n_y$. The blur corresponds to a convolution with a known low-pass kernel. Let $A : \mathbb{R}^{n_x \times n_y} \to \mathbb{R}^{n_x \times n_y}$ be the blur linear operator. We aim to solve the (linear) inverse problem of reconstructing $u^\star \in \mathbb{R}^{n_x \times n_y}$ from the relation $b = A\bar{u} + \xi$, where ξ is the noise, that is additive pixel-wise, has $0-$mean and is Gaussian. Through this experiment, we used $n_x = n_y = 256$.

In order to reduce noise amplification when inverting the operator A, we solve a regularized optimization problem to recover $u^\star$ as accurately as possible. As natural images can be assumed to be smooth except for a (small) edge-set between objects in the image, we use a non-convex logarithmic regularization term that penalizes finite forward differences in horizontal and vertical directions of the image, implemented as linear operators $K_x, K_y : \mathbb{R}^{n_x \times n_y} \to \mathbb{R}^{n_x \times n_y}$ with Neumann boundary conditions. In summary, we aim to solve the following:

$$\min_{u \in \mathbb{R}^{n_x \times n_y}} f(u), \quad f(u) \stackrel{\text{def}}{=} \frac{1}{2}\|Au - b\|^2 + \frac{\mu}{2}\sum_{i=1}^{n_x}\sum_{j=1}^{n_y} \log(\rho + (K_x u)_{i,j}^2 + (K_y u)_{i,j}^2),$$

where μ, ρ are positive constants for regularization and numerical stability set to $5 \cdot 10^{-5}$ and 10^{-3}, respectively. f definable as the sum of compositions of definable mappings, and ∇f is Lipschitz continuous.

To solve the above optimization problem, we have used (ISEHD-Disc) and (ISIHD-Disc) with parameters $\beta = 1.3, \gamma_k \equiv 0.25, h = 0.5$, and initial conditions $x_0 = x_1 = 0_{n_x \times n_y}$. We compared both algorithms with the baseline algorithms (GD), (HBF) (with the same initial condition). All algorithms were run for 250 iterations. The results are shown in Figure 3 and Figure 4.

In Figure 3, the original image $\bar{u}$ is shown on the left. In the middle, we display the blurry and noise image b. Finally, the image recovered by (ISEHD-Disc) is shown on the right.

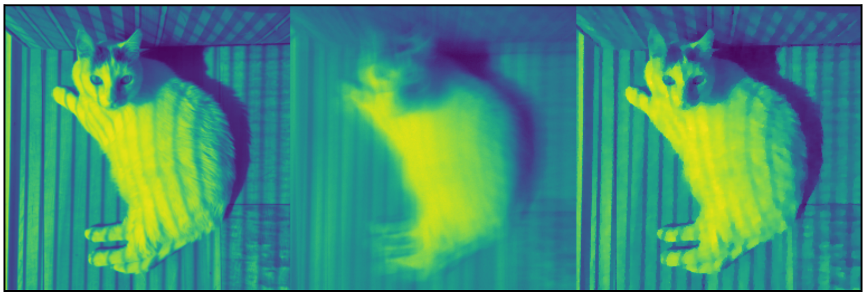
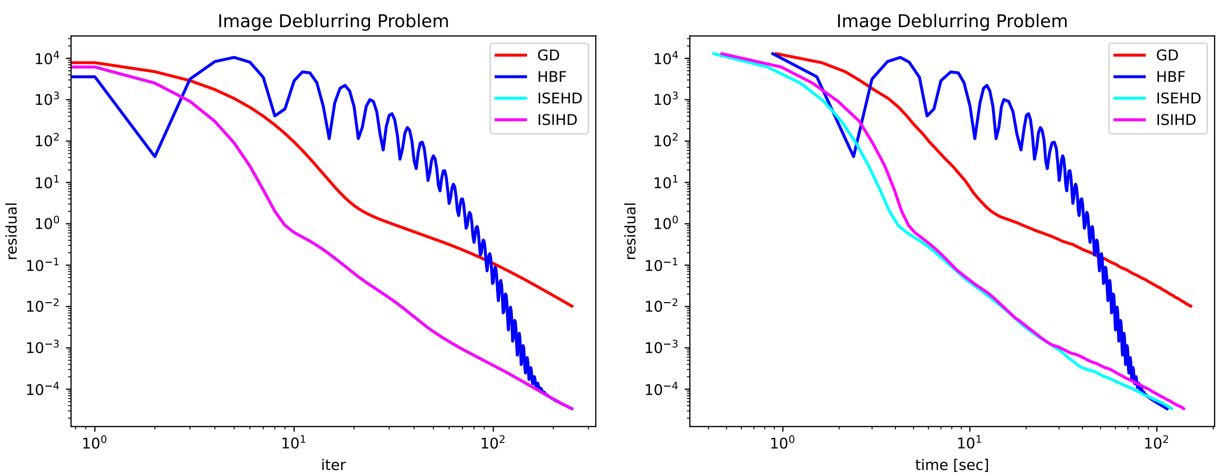


Figure 3: Results of the discretization of ISEHD.

In Figure 4, see that the residual plots of (ISEHD-Disc) and (ISIHD-Disc) overlap. Again, as expected, the trajectory of (ISEHD-Disc) and (ISIHD-Disc) has much less oscillation than (HBF) which is a very desirable feature in practice. At the same time, (ISEHD-Disc) and (ISIHD-Disc) see to convergence faster, though (HBF) eventually shows a similar convergence rate. Again, (GD) is the slowest. Overall, (ISEHD-Disc) and (ISIHD-Disc) seem to take the best of both worlds: small oscillations and a faster asymptotic convergence rate.



(a) Iteration vs Residual.

(b) Time vs Residual.

Figure 4: Results on the image deblurring problem.

6 Conclusion and Perspectives

We conclude that:

- Under definability conditions on the objective and suitable conditions on γ, we obtain convergence of the trajectory of (ISEHD) and (ISIHD). Besides, in the autonomous setting, and under a Morse condition, the trajectory almost surely converges to a local minimum of the objective.
- We obtain analogous properties for the respective proposed algorithmic schemes (ISEHD-Disc) and (ISIHD-Disc).

- The inclusion of the term β helps to reduce oscillations towards critical points, and, when chosen appropriately, without reducing substantially the speed of convergence of the case $\beta = 0$, *i.e.* the one for Heavy Ball with Friction method.
- The selection of β is important. If it is chosen too close to zero, it may not significantly reduce oscillations. Conversely, if it is chosen too large (even within theoretical bounds), the trade-off for reduced oscillations might be a worse convergence rate.

Several open problems are worth investigating in the future:

- Replacing the global Lipschitz continuity assumption on the gradient in Theorems 3.7 and 4.5 with a local Lipschitz continuity assumption as proposed in [67, 68, 81].
- Proposing discrete schemes of (ISEHD) and (ISIHD) with a variable stepsize, which can be computed, for instance, by backtracking.
- Extending our results to (non-euclidian) Bregman geometry.
- Extending our results to the non-smooth setting as proposed in [40].

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