Bregman Stochastic Proximal Point Algorithm with Variance Reduction

Cheik Traoré[✉] [*]   Peter Ochs[†]

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Abstract

Stochastic algorithms, especially stochastic gradient descent (SGD), have proven to be the go-to methods in data science and machine learning. In recent years, the stochastic proximal point algorithm (SPPA) emerged, and it was shown to be more robust than SGD with respect to stepsize settings. However, SPPA still suffers from a decreased convergence rate due to the need for vanishing stepsizes, which is resolved by using variance reduction methods. In the deterministic setting, there are many problems that can be solved more efficiently when viewing them in a non-Euclidean geometry using Bregman distances. This paper combines these two worlds and proposes variance reduction techniques for the Bregman stochastic proximal point algorithm (BSPPA). As special cases, we obtain SAGA- and SVRG-like variance reduction techniques for BSPPA. Our theoretical and numerical results demonstrate improved stability and convergence rates compared to the vanilla BSPPA with constant and vanishing stepsizes, respectively. Our analysis, also, allow to recover the same variance reduction techniques for Bregman SGD in a unified way.

Key words and phrases. Stochastic proximal point algorithm, stochastic gradient descent, variance reduction, Bregman distance, convex optimization, smooth optimization.

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1 Introduction

The objective of the paper is to solve the following finite-sum optimization problem

$$\underset{\mathsf{x}\in\mathsf{H}}{\text{minimize}}\ \frac{1}{n}\sum_{i=1}^{n} f_i(\mathsf{x}), \tag{1.1}$$

where $\mathsf{H} \subset \mathbb{R}^d$ is closed and convex with non empty interior, and $f_i\colon \mathbb{R}^d \to \mathbb{R}\cup\{+\infty\}$ is convex, for $i \in \{1,\dots,n\}$.

The prime example of Problem (1.1) is the Empirical Risk Minimization (ERM) problem in machine learning [28, Section 2.2]. In that setting, n is the number of data points, $\mathsf{x} \in \mathbb{R}^d$ includes the parameters of a machine learning model (linear functions, neural networks, etc.),

[*]Corresponding author: Toulouse School of Economics, Toulouse Capitole University, Toulouse, France (cheik.traore@tse-fr.eu).

[†]Department of Mathematics and Computer Science, Saarland University, Saarbrücken, Germany (ochs@cs.uni-saarland.de).

and the function f_i is the loss of the model x at the i-th data point. To efficiently solve Problem (3.1), stochastic approximation [26] is leveraged, in particular stochastic gradient descent (SGD) and its variants [12, 19].

Stochastic Proximal Point Algorithm. In recent years, the stochastic proximal point algorithm (SPPA) [1, 4, 18, 23, 27, 29–31] has emerged as a good alternative to SGD, demonstrating greater robustness to stepsize selection. Instead of the gradient ∇f_i, the proximity operator of f_i, chosen randomly, is used at each iteration. Davis et al. [7] studied a Bregman distance version of SPPA (BSPPA) to account for a non-Euclidean geometry that can be adapted to the problem. While the convergence rates of both SPPA and BSPPA are of the same orders as those of SGD, in the deterministic setting, well adapted Bregman based algorithms can improve the constants in the rates significantly (e.g. from linear to logarithmic dependence on the problem dimension [22]). We trace this dis-balance between the stochastic and deterministic setting back to the variance of the stochastic estimator of the true proximity operator that requires vanishing stepsize for the algorithm to converge.

Variance-reduced Stochastic Proximal Point Algorithms have only recently emerged [16, 21, 25, 32]. As in this paper, all existing convergence results are provided in the smooth (differentiable) case, except for Point-SAGA proposed by Defazio [8] and its generalization SMPM proposed by Condat et al. [6], where sublinear convergence rates ($\mathcal{O}(1/k)$ and $\mathcal{O}(1/k^2)$ respectively) with a constant stepsize are provided for strongly convex functions. However, to the best of our knowledge, no existing result has been established in the Bregman setting. The latter can be better adapted to constrained optimization problems or to cases where the Euclidean distance fails to adequately capture the underlying properties.

Contributions. In this paper we propose a Bregman Stochastic Proximal Point Algorithm (BSPPA) with generic variance reduction. We provide improved convergence rates as compared with vanilla BSPPA without variance reduction, in particular, sublinear and linear rates for convex or relatively strongly convex functions, respectively. Then, from the generic results, convergence rates for BSAPA, BSVRP, and L-BSVRP, which are proximal and Bregman distance versions of SAGA [9], SVRG [15], and L-SVRG [20] respectively. These variance-reduced algorithms are all new in the literature. We can also recover rates for vanilla BSPPA for potentially non-relatively smooth functions. Of course, the rates of the variance-reduced versions (BSAPA, BSVRP, L-BSVRP) are given without the need for vanishing stepsizes, and are faster than that of BSPPA, which are the intended objectives of variance reduction. As a by-product of our study, we provide a unified variance reduction analysis for Bregman SGD. Our analysis can thus recover not only the Bregman version of SAGA and L-SVRG (present in Dragomir et al. [11]), but also Bregman SVRG and the Bregman version of the unified study in Gorbunov et al. [13], dealing with several variants of (variance-reduced) Euclidean SGD.

2 Preliminaries

2.1 Notation

Let $\mathcal{C} \subset \mathbb{R}^d$ be convex with non empty interior. We denote interior of $\mathcal{C}$ by $\operatorname{int}\mathcal{C}$ and its closure by $\bar{\mathcal{C}}$. For every integer $\ell \geq 1$, we define $[\ell] := \{1, \ldots, \ell\}$. Bold default font is used for random variables taking values in $\mathbb{R}^d$, while bold sans serif font is used for their realizations or deterministic variables in $\mathbb{R}^d$. The probability space underlying random variables is denoted by $(\Omega, \mathfrak{A}, \mathsf{P})$. For every random variable x, $\mathsf{E}[x]$ denotes its expectation, while if $\mathfrak{F} \subset \mathfrak{A}$ is a sub σ-algebra we denote by $\mathsf{E}[x \,|\, \mathfrak{F}]$ the conditional expectation of x given $\mathfrak{F}$. Also, $\sigma(y)$ represents the σ-algebra generated by the random variable y. For a function $\varphi\colon \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$, define $\operatorname{dom}\varphi := \left\{\mathsf{x} \in \mathbb{R}^d : \varphi(\mathsf{x}) < +\infty\right\}$. φ is proper if $\operatorname{dom}\varphi \neq \varnothing$. The set of minimizers of φ is $\arg\min \varphi = \{\mathsf{x} \in \mathbb{R}^d : \varphi(\mathsf{x}) = \inf \varphi\}$. If $\inf \varphi$ is finite, it is represented by φ_*. We denote the subdifferential of φ at x as $\partial\varphi(\mathsf{x}) := \left\{\mathsf{g} \in \mathbb{R}^d : \varphi(\mathsf{y}) - \varphi(\mathsf{x}) \geq \langle \mathsf{g}, \mathsf{y} - \mathsf{x}\rangle \ (\forall\, \mathsf{y} \in \mathbb{R}^d)\right\}$. When φ is differentiable $\nabla\varphi$ denotes the gradient of φ. The (convex) conjugate of φ is the function $\varphi^*\colon \mathbb{R}^d \to [-\infty, +\infty]$ defined by $\mathsf{y} \mapsto \sup_{\mathsf{x}\in\mathbb{R}^d} \langle \mathsf{y}, \mathsf{x}\rangle - \varphi(\mathsf{x})$. In this work, ℓ^1 represents the space of sequences with summable norms and ℓ^2 with summable squared norms.

2.2 Bregman distance

The Bregman distance $D_h(\mathsf{x}, \mathsf{y})$, between $\mathsf{x} \in \mathcal{C}, \mathsf{y} \in \operatorname{int}\mathcal{C}$, is defined as $D_h(\mathsf{x}, \mathsf{y}) := h(\mathsf{x}) - h(\mathsf{y}) - \langle \nabla h(\mathsf{y}), \mathsf{x} - \mathsf{y}\rangle$, where $h\colon \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$, often called kernel, is a strictly convex and twice continuously differentiable function on $\operatorname{int}\mathcal{C}$, with $\operatorname{dom} h = \mathcal{C}$. In what follows, h will always refer to a function with those properties, unless stated otherwise. A classical example is the usual squared Euclidean distance $D_h(\mathsf{x}, \mathsf{y}) = \frac{1}{2}\|\mathsf{x} - \mathsf{y}\|_2^2$ when $h = \frac{1}{2}\|\cdot\|_2^2$. The Bregman distance often captures the underlying geometry of an optimization problem more effectively than the Euclidean distance. It can be better suited to the constraints set or to the objective function properties. This makes it a powerful tool in optimization. For instance, the Kullback–Leibler divergence on $\mathbb{R}^d$ does not have a global Lipschitz continuous gradient, however it is relatively smooth w.r.t. the kernel $h(\mathsf{x}) = -\sum_i \log \mathsf{x}_i$; see Bauschke et al. [3]. Unfortunately, the Bregman distance is not necessarily symmetric, homogeneous nor translation invariant. The latter two shortcomings are the reasons why the following assumption is required.

Assumptions 2.1. *Let $\mathsf{x}, \mathsf{y} \in \operatorname{int}\operatorname{dom} h^*$, $\lambda \in \mathbb{R}$, and $\mathsf{z} \in \mathbb{R}^d$ such that $\mathsf{x} + \lambda\mathsf{z}, \mathsf{y} + \mathsf{z} \in \operatorname{dom} h^*$. There exists a positive gain function G such that*

$$D_{h^*}(\mathsf{x} + \lambda\mathsf{z}, \mathsf{x}) \leq G(\mathsf{x}, \mathsf{y}, \mathsf{z})\lambda^2 D_{h^*}(\mathsf{y} + \mathsf{z}, \mathsf{y}), \tag{2.1}$$

According to Dragomir et al. [11], this assumption seems unavoidable when using past iterates in an algorithm with Bregman distance. This is also the case of accelerated methods where similar assumption is made; see Hanzely et al. [14].

Proposition 2.2. [11, Proposition 1]. *If h is L-smooth and the Hessian $\nabla^2 h^*$ is M-smooth, then the gain function can be chosen as:*

$$G(\mathsf{x}, \mathsf{y}, \mathsf{v}) = 1 + 2ML(\|\mathsf{y} - \mathsf{x}\| + \|\mathsf{v}\|).$$

To address the symmetry issue, as Bauschke et al. [3], we will make use of the following symmetry coefficient. It will only be needed in the relatively strongly convex cases.

Definition 2.3. Given a kernel $h\colon \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$, its symmetry coefficient is defined by

$$\gamma_h := \inf\left\{\frac{D_h(\mathsf{x}, \mathsf{y})}{D_h(\mathsf{y}, \mathsf{x})} : (\mathsf{x}, \mathsf{y}) \in (\operatorname{int}\mathcal{C})^2, \mathsf{x} \neq \mathsf{y}\right\} \in [0, 1].$$

Remark 2.4.

1. From Bauschke et al. [2, Theorem 3.7], we know that $D_h(\mathsf{x}, \mathsf{y}) = D_{h^*}(\nabla h(\mathsf{y}), \nabla h(\mathsf{x}))$ for all $(\mathsf{x}, \mathsf{y}) \in (\operatorname{int}\,\mathcal{C})^2$. Hence, $\gamma_h = \gamma_{h^*}$.

2. By definition, $(\forall \mathsf{x} \in \operatorname{int} \mathcal{C})$ $(\forall \mathsf{y} \in \operatorname{int} \mathcal{C})$,

$$\gamma_h D_h(\mathsf{y}, \mathsf{x}) \leq D_h(\mathsf{x}, \mathsf{y}) \leq \gamma_h^{-1} D_h(\mathsf{y}, \mathsf{x}),$$

where it is agreed that $0^{-1} = +\infty$ and $+\infty \times r = +\infty$ for all $r \geq 0$.

Next, we present a handy identity for Bregman distances.

Lemma 2.5 (Three points identity [5])**.** *Let $h \colon \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semi-continuous convex function. For any $\mathsf{x} \in \operatorname{dom} h$, and $\mathsf{y}, \mathsf{z} \in \operatorname{int} \operatorname{dom} h$ the following identity holds:*

$$D_h(\mathsf{x}, \mathsf{z}) - D_h(\mathsf{x}, \mathsf{y}) - D_h(\mathsf{y}, \mathsf{z}) = \langle \nabla h(\mathsf{y}) - \nabla h(\mathsf{z}), \mathsf{x} - \mathsf{y} \rangle.$$

3 Problem setting

First, we (re)introduce the finite-sum problem and collect all assumptions that we need for this paper, while not all of them will be required to hold at the same time. Section 4 presents the proposed algorithm for solving such problems:

$$\underset{\mathsf{x} \in \mathsf{H}}{\text{minimize}} \; F(\mathsf{x}), \quad F(\mathsf{x}) := \frac{1}{n} \sum_{i=1}^{n} f_i(\mathsf{x}), \tag{3.1}$$

where $\mathsf{H} := \bar{\mathcal{C}}$ is the closure of $\mathcal{C}$ and $f_i \colon \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is proper, convex and lower semicontinuous (lsc), $i \in \{1, 2, \ldots, n\}$.

Assumptions 3.1. (A.i) *$\mathcal{C} \subset \operatorname{dom} F$ and $\mathsf{x}_* \in \mathcal{C}$ solves* (3.1)*.*

(A.ii) *For all $i \in [n]$, f_i is β-relatively strongly convex w.r.t. h, with $\beta \geq 0$, i.e.,*

$$(\forall \mathsf{x}, \mathsf{y} \in \operatorname{int} \mathcal{C}) \quad \beta D_h(\mathsf{x}, \mathsf{y}) \leq D_{f_i}(\mathsf{x}, \mathsf{y}).$$

It includes the standing convexity assumption of each f_i, i.e. $\beta = 0$.

(A.iii) *For all $i \in [n]$, f_i is L-relatively smooth w.r.t. to h, i.e., differentiable on* $\operatorname{int} \mathcal{C}$ *and*

$$(\forall \mathsf{x}, \mathsf{y} \in \operatorname{int} \mathcal{C}) \quad D_{f_i}(\mathsf{x}, \mathsf{y}) \leq L D_h(\mathsf{x}, \mathsf{y}),$$

for some $L > 0$. As a consequence, F is L-relatively smooth.

(A.iv) *F is μ-relatively strongly convex w.r.t. h, with $\mu > 0$.*

(A.v) $\mathsf{x}_* \in \operatorname{int} \mathcal{C}$ *such that* $\nabla F(\mathsf{x}_*) = 0$.

Assumption (A.i) is always supposed true in this paper. Assumptions (A.ii) and (A.iv) extend the Euclidean notions of strong convexity and L-smoothness (i.e., L-Lipschitz continuity of the gradient), respectively, to the Bregman setting. Assumption (A.ii), with $\beta > 0$, is only required when establishing a linear convergence rate (to a neighborhood of the optimum) for the vanilla BSPPA algorithm. Similarly, (A.iv), which corresponds to the relative strong convexity of F, will be used to derive linear convergence rates to the exact minimum for the variance-reduced variants of BSPPA; in that case, only convexity of each f_i will be assumed. Finally, (A.iv) and (A.ii) with $\beta > 0$ are not required to obtain the sublinear convergence rates. Regarding Assumption (A.v), our main focus is not on the constrained cases where it may fail. Instead, we aim to address, possibly unconstrained, problems whose objective functions lack Lipschitz-continuous gradients—rendering standard Euclidean algorithms unsuitable—but are relatively smooth with respect to a Bregman kernel. Assumption (A.v) will be stated whenever needed.

4 BSPPA with generic variance reduction

We propose to solve Problem (3.1) using the following Bregman stochastic proximal point algorithm (BSPPA) with a generic variance reduction satisfying the abstract conditions in Assumption 4.5 further below.

Algorithm 4.1. Let $(e_k)_{k\in\mathbb{N}}$ be a sequence of random vectors in $\mathbb{R}^d$ and let $(i_k)_{k\in\mathbb{N}}$ be a sequence of i.i.d. random variables uniformly distributed on $\{1,\ldots,n\}$, so that i_k is independent of $e_0, \ldots, e_{k-1}$. Let $\alpha_k > 0$ and set $x_0 \equiv \mathsf{x}_0 \in \operatorname{int}\mathcal{C}$. Then define,

$$
\begin{array}{l}
\text{for } k = 0, 1, \ldots \\
\left\lfloor \; x_{k+1} \quad = \arg\min_{\mathsf{x}\in\mathsf{H}} \left\{ f_{i_k}(\mathsf{x}) - \langle e_k, \mathsf{x} - x_k\rangle + \frac{1}{\alpha_k} D_h(\mathsf{x}, x_k) \right\}. \right.
\end{array}
$$

We assume in all this work that this minimization is well-posed, meaning there exists a unique solution in $\operatorname{int}\mathcal{C}$. Compared to BSPPA, we have an additional linear perturbation that contains e_k. It is the generic variance reduction term. As we shall see in Section 5, depending on the specific algorithms, e_k may be defined in various ways. When e_k is set to 0, BSPPA is recovered. Using the optimality condition, the update at iteration k can be rewritten as:

$$
x_{k+1} = \nabla h^*\Big(\nabla h(x_k) - \alpha_k \underbrace{(g_{k+1} - e_k)}_{=:w_k}\Big), \tag{4.1}
$$

where $g_{k+1} \in \partial f_{i_k}(x_{k+1})$ is such that (4.1) holds. Since w_k depends on g_{k+1}, Equation (4.1) shows that Algorithm 4.1 is an implicit algorithm. This is in contrast to an explicit update

$$
z_{k+1} = \nabla h^*\left(\nabla h(x_k) - \alpha_k v_k\right) \tag{4.2}
$$

in which w_k is replaced by

$$
v_k \coloneqq g_k - e_k \tag{4.3}
$$

for a suitable $g_k \in \partial f_{i_k}(x_k)$. In minimization form, this explicit update reads

$$
z_{k+1} = \operatorname*{arg\,min}_{\mathsf{x}\in\mathsf{H}} \langle v_k, \mathsf{x} - x_k\rangle + \frac{1}{\alpha_k} D_h(\mathsf{x}, x_k), \tag{4.4}
$$

and is assumed well-posed, i.e., a unique solution in $\operatorname{int}\mathcal{C}$ exists for any $k \in \mathbb{N}$.

Remark 4.2. The virtual explicit sequences $(v_k)_{k\in\mathbb{N}}$ and $(z_k)_{k\in\mathbb{N}}$ introduced in (4.2) and (4.3) are crucial to our analysis. They will be used to prove the main proposition even though they do not appear in Algorithm 4.1.

Remark 4.3 (Bregman variance-reduced SGD also covered). The integration of these explicit iterates is the reason why our analysis extends seamlessly to the Bregman SGD case, providing a unified variance reduction study for Bregman SGD. All the variance reduction results will stand true for SGD with Bregman distance; see Remark B.2 and Proposition B.3.

The next assumptions that we consider concern the noise and variance of Algorithm 4.1. The virtual explicit sequence $(z_k)_{k\in\mathbb{N}}$ appears in those assumptions for the sake of analysis. This allows, at the same time, the application of the analysis and the results to the Bregman SGD case.

Remark 4.4. These assumptions on the noise and variance of the generic Algorithm 4.1 are "assumed" for the general analysis. But when algorithms are specified in Section 5 by defining e_k, they will be proved by corresponding lemmas. They read as follows.

Assumptions 4.5. *There exist non-negative sequences of real numbers* $(A_k)_{k\in\mathbb{N}}, (B_k)_{k\in\mathbb{N}}, (C_k)_{k\in\mathbb{N}}$, $\rho \in [0,1]$, *and a sequence of real-valued random variables* $(N_k)_{k\in\mathbb{N}}$ *such that, for every* $k \in \mathbb{N}$,

(B.i) $\mathsf{E}[e_k \,|\, \mathfrak{F}_k] = 0$ *a.s. and*
$\mathsf{E}[g_k \,|\, \mathfrak{F}_k] \in \partial F(x_k)$ *a.s.,*

(B.ii) $\mathsf{E}[D_h(x_k, z_{k+1}) \,|\, \mathfrak{F}_k] \leq \alpha_k^2 A_k \left(F(x_k) - F(\mathsf{x}_*)\right) + \alpha_k^2 B_k \sigma_k^2 + \alpha_k^2 N_k$ *a.s.,*

(B.iii) $\mathsf{E}[\sigma_{k+1}^2] \leq (1-\rho)\mathsf{E}\left[\sigma_k^2\right] + C_k \mathsf{E}[F(x_k) - F(\mathsf{x}_*)]$,

where σ_k *is a real-valued random variable (r.v.),* $(\mathfrak{F}_k)_{k\in\mathbb{N}}$ *is a sequence of* σ*-algebras such that,* $\forall k \in \mathbb{N}$, $\mathfrak{F}_k \subset \mathfrak{F}_{k+1} \subset \mathfrak{A}$; i_{k-1}, x_k, σ_k^2 *and* N_k *are* $\mathfrak{F}_k$*-measurable, and* i_k *is independent of* $\mathfrak{F}_k$.

In the smooth case, Assumption (B.i) ensures that $\mathsf{E}[v_k \,|\, \mathfrak{F}_k] = \mathsf{E}[\nabla f_{i_k}(x_k) \,|\, \mathfrak{F}_k] = \nabla F(x_k)$, so that the direction v_k is an unbiased estimator of the full gradient of F at x_k, which is a standard assumption in the related literature. Assumption (B.ii) on $\mathsf{E}\left[D_h(x_k, z_{k+1}) \,|\, \mathfrak{F}_k\right]$ is the equivalent of what is called, in the literature [10, 17] and in the Euclidean setting, the expected smoothness or ABC-assumption on $\mathsf{E}\left[\|\nabla f_{i_k}(x_k)\|^2 \,|\, \mathfrak{F}_k\right]$ with $\sigma_k = \|\nabla F(x_k)\|$ and N_k constant. Assumption (B.iii) provides some control on the variance from iteration to iteration. Indeed, as we will see later that the sequence $(\sigma_k)_{k\in\mathbb{N}}$ encodes the variance of the Algorithm 4.1. Depending on ρ, (B.iii) makes sure that the variance does not blow up and possibly reduces along the iterations whenever the algorithm converges.

4.1 Convergence Analysis

Now, we can present the main two theorems.

Theorem 4.6 (F is only convex)**.** *Suppose that Assumptions 4.5 hold with* $\rho > 0$. *Let* $(M_k)_{k\in\mathbb{N}}$ *be a non-increasing positive real-valued sequence such that* $M_k \geq \frac{B_k}{\rho}$, $\forall k \in \mathbb{N}$. *Suppose also that the sequence* $(x_k)_{k\in\mathbb{N}}$ *is generated by Algorithm 4.1 with* $(\alpha_k)_{k\in\mathbb{N}}$ *a non-decreasing positive real-valued sequence such that* $\alpha_k < \frac{1}{A_k + M_k C_k}$, $\forall k \in \mathbb{N}$. *Then, for all* $k \in \mathbb{N}$,

$$\mathsf{E}[F(\bar{x}_k) - F(\mathsf{x}_*)] \leq \frac{(1/\alpha_0^2)\mathsf{E}[D_h(\mathsf{x}_*, x_0)] + M_0 \mathsf{E}[\sigma_0^2]}{\sum_{t=0}^{k-1}(1/\alpha_t)\left(1 - \alpha_t(A_t + M_t C_t)\right)} + \sum_{t=0}^{k-1} \frac{\mathsf{E}[N_t]}{\sum_{t=0}^{k-1}(1/\alpha_t)\left(1 - \alpha_t(A_t + M_t C_t)\right)},$$

with $\bar{x}_k = \displaystyle\sum_{t=0}^{k-1} \frac{(1/\alpha_t)\left(1 - \alpha_t(A_t + M_t C_t)\right)}{\sum_{t=0}^{k-1}(1/\alpha_t)\left(1 - \alpha_t(A_t + M_t C_t)\right)} x_t.$

Theorem 4.7 (F is μ-relatively strongly convex)**.** *Suppose that Assumptions* (A.v), *4.5 and* (A.iv) *are verified with* $\rho > 0$. *Let* $(M_k)_{k\in\mathbb{N}}$ *be a non-increasing positive real-valued sequence such that* $M_k > \frac{B_k}{\rho}$, $\forall k \in \mathbb{N}$. *Suppose also that the sequence* $(x_k)_{k\in\mathbb{N}}$ *is generated by Algorithm 4.1 with* $(\alpha_k)_{k\in\mathbb{N}}$ *a non-decreasing positive real-valued sequence such that* $\alpha_k < \frac{1}{A_k + M_k C_k}$, $\forall k \in \mathbb{N}$. *Set* $q_k := \max\left\{1 - \alpha_k \gamma_h \mu \left(1 - \alpha_k(A_k + M_k C_k)\right), 1 + \frac{B_k}{M_k} - \rho\right\}$. *Then for all* $k \in \mathbb{N}$, $q_k \in]0,1[$ *and*

$$V_{k+1} \leq q_k V_k + \mathsf{E}[N_k],$$

where $V_k = \mathsf{E}\left[\dfrac{1}{\alpha_k^2} D_h(\mathsf{x}_*, x_k) + M_k \sigma_k^2\right]$, $\forall k \in \mathbb{N}$.

Remark 4.8. In both Theorem 4.6 and 4.7, on the right hand sides, the second terms may seem problematic. However, later, for specified variance-reduced algorithms, we will have $N_k \le 0$ for every $k \in \mathbb{N}$. So they can be dropped. Therefore, in terms of order of convergence, these theorems establish the standard sublinear $\mathcal{O}(1/k)$ and linear $\mathcal{O}(q^k)$ rates for the generic Algorithm 4.1, corresponding, respectively, to convex and relatively strongly convex functions, without requiring a vanishing stepsize. This is an improvement on BSPPA without variance reduction; see Section 5.1. However, the constants in both theorems depend on the sequence $(G_k)_{k\in\mathbb{N}}$ and this can impede those rates. In the ideal case of Euclidean or quadratic kernel, $G_k = 1$ [11], and we recover the generic results of Traoré et al. [32].

5 Instantiation of specific algorithms

In this section, we specialize the generic Algorithm 4.1 with different variance reduction techniques by specifying the term e_k.

5.1 Bregman Stochastic Proximal Point Algorithm (BSPPA)

We start by the vanilla BSPPA [7], which is also covered by the generic algorithm and analysis by taking $e_k = 0$ for all $k \in \mathbb{N}$. For BSPPA, the functions can potentially be nonsmooth.

Algorithm 5.1 (BSPPA). Let $(i_k)_{k\in\mathbb{N}}$ be a sequence of i.i.d. random variables uniformly distributed on $\{1, \dots, n\}$. Let $x_0 \equiv \mathsf{x}_0 \in \operatorname{int}\mathcal{C}$ and $\alpha_k > 0$ for all $k \in \mathbb{N}$.

$$\begin{array}{l} \text{for } k = 0, 1, \dots \\ \left\lfloor \; x_{k+1} = \arg\min_{\mathsf{x}\in\mathsf{H}}\{f_{i_k}(\mathsf{x}) + \frac{1}{\alpha_k} D_h(\mathsf{x}, x_k)\}. \right. \end{array}$$

Theorem 5.2 (F is only convex)**.** *Suppose that Assumptions* (B.i) *and* (B.ii) *hold with $A_k = N_k = 0$ and $B_k\sigma_k^2 \le \sigma_*^2 \ge 0$ (i.e., $\mathsf{E}[D_h(x_k, z_{k+1}) \,|\, \mathfrak{F}_k]$ is bounded by $\alpha_k^2\sigma_*^2$). Suppose also that the sequence $(x_k)_{k\in\mathbb{N}}$ is generated by Algorithm 5.1. Then, for $k \ge 1$,*

$$\mathsf{E}[F(\bar{x}_k) - F(\mathsf{x}_*)] \le \frac{D_h(\mathsf{x}_*, x_0)}{\sum_{t=0}^{k-1}\alpha_t} + \sigma_*^2 \frac{\sum_{t=0}^{k-1}\alpha_t^2}{\sum_{t=0}^{k-1}\alpha_t},$$

where $\bar{x}_k = \displaystyle\sum_{t=0}^{k-1} \frac{\alpha_t}{\sum_{t=0}^{k-1}\alpha_t} x_t$.

Remark 5.3. Here, σ_*^2 represents a hard bound on $\mathsf{E}[D_h(x_k, z_{k+1}) \,|\, \mathfrak{F}_k]$ and, in some sense, encodes the variance of the algorithm. If we set the stepsize to be constant, i.e. $\alpha_k = \alpha$, we obtain

$$\mathsf{E}[F(\bar{x}_k) - F(\mathsf{x}_*)] \le \frac{D_h(\mathsf{x}_*, x_0)}{\alpha k} + \alpha\sigma_*^2.$$

This equation shows that the algorithm, because of the variance, will converge to a ball around the minimum rather than the minimum itself and keep oscillating. A vanishing stepsize (typically $(\alpha_k)_{k\in\mathbb{N}} \in \ell_2 \setminus \ell_1$) can be used in order to cancel σ_*^2 asymptotically. But, this slows down the algorithm to a convergence of order $O(1/\sqrt{k})$.

Theorem 5.4. *Suppose that Assumptions* (A.ii)*,* (B.i)*, and* (B.ii) *hold with $\beta > 0$ (each f_i is relatively strongly convex), $A_k = N_k = 0$ and $B_k\sigma_k^2 \le \sigma_*^2 \ge 0$ (i.e., $\mathsf{E}[D_h(x_k, z_{k+1}) \,|\, \mathfrak{F}_k]$*

is bounded by $\alpha_k^2\sigma_*^2$). Let $\alpha_k = \alpha > 0$, $\forall k \in \mathbb{N}$. Suppose also that the sequence $(x_k)_{k\in\mathbb{N}}$ is generated by Algorithm 5.1. Then, for $k \geq 1$,

$$\mathsf{E}[D_h(\mathsf{x}_*, x_k)] \leq q^k \mathsf{E}[D_h(\mathsf{x}_*, x_0)] + \alpha^2 \frac{1}{1-q}\sigma_*^2,$$

where $q = \frac{1}{1+\beta\alpha}$.

Remark 5.5. As in the only convex case, with a constant stepsize, the algorithm will only converge to a ball around the minimizer and oscillate there due to the variance. Also using a vanishing to kill the variance leads to reduce convergence rate from linear to $O(1/k)$. So we need another way to reduce the variance, that what the next sections will present.

Remark 5.6 (relatively smooth and interpolation case)**.** If f_i is relatively smooth for all $i \in [n]$, $B_k\sigma_k^2 \propto \mathsf{E}_k[D_{h^*}(\nabla h(x_k) - 2\alpha\nabla f_{i_k}(\mathsf{x}_*), \nabla h(x_k))]$; see Dragomir et al. [11, Section 3.2]. Therefore, in the very special case of interpolation, i.e. $\nabla f_i(\mathsf{x}_*) = 0$ for all $i \in [n]$, $\sigma_*^2 = 0$ and BSPPA does converge to the actual minimum and have a good sublinear and linear rates for convex and relatively strongly convex functions, respectively.

In the following, we will present the variance-reduced algorithms. We defer the double-loop, SVRG-style variant of the variance-reduced BSPPA to Appendix A, and focus in the main text on its single-loop counterpart. The first variance-reduced BSPPA that we proposed is using a SAGA-style technique, and we coined it BSAPA. It is the Bregman version of SAPA proposed by Traoré et al. [32].

5.2 Bregman SAPA (BSAPA)

Algorithm 5.7 (**BSAPA**)**.** Let $(i_k)_{k\in\mathbb{N}}$ be a sequence of i.i.d. random variables uniformly distributed on $\{1, \dots, n\}$. Let $\alpha_k > 0$ for every $k \in \mathbb{N}$, and set, for every $i \in [n]$, $x_0 \equiv \phi_i^0 \equiv \mathsf{x}_0 \in \operatorname{int} \mathcal{C}$.

for $k = 0, 1, \dots$

$$\left\lfloor \begin{array}{l} x_{k+1} = \arg\min_{\mathsf{x}\in\mathsf{H}} \left\{ f_{i_k}(\mathsf{x}) + \frac{1}{\alpha_k} D_h(\mathsf{x}, x_k) - \langle \nabla f_{i_k}(\phi_{i_k}^k) - \frac{1}{n}\sum_{i=1}^n \nabla f_i(\phi_i^k), \mathsf{x} - x_k \rangle \right\} \\ \forall\, i \in [n]\colon\ \phi_i^{k+1} = \phi_i^k + \delta_{i,i_k}(x_k - \phi_i^k), \end{array} \right.$$

where $\delta_{i,j}$ is the Kronecker symbol. Here we get $e_k = \nabla f_{i_k}(\phi_{i_k}^k) - \frac{1}{n}\sum_{i=1}^n \nabla f_i(\phi_i^k)$. We set $\mathfrak{F}_k = \sigma(i_0, \dots, i_{k-1})$ and $\mathsf{E}_k[\cdot] = \mathsf{E}[\cdot \,|\, \mathfrak{F}_k]$. We then have that x_k and ϕ_i^k are $\mathfrak{F}_k$-measurable and i_k is independent of $\mathfrak{F}_k$. Let $\zeta_k = -2\alpha_k(\nabla f_{i_k}(\mathsf{x}_*) - \nabla f_{i_k}(\phi_{i_k}^k))$. It is clear that $\mathsf{E}_k[\zeta_k] = 2\alpha_k \frac{1}{n}\sum_{i=1}^n \nabla f_i(\phi_i^k)$.

Lemma 5.8. *Suppose that Assumptions* (A.iii)*, 2.1 and* (A.v) *hold. Let $s \in \mathbb{N}$ and let $(x_k)_{k\in[m]}$ be the sequence generated by the inner iteration in Algorithm 5.13. We finally assume that there exists a non-increasing sequence $(G_k)_{k\in\mathbb{N}}$ such that, for all $i \in [n]$,*

$$G_k \geq G\left(\nabla h(x_k), \nabla h(x_k), \frac{1}{L}(\nabla f_i(x_k) - \nabla f_i(\mathsf{x}_*))\right), \tag{5.1}$$

$$G_k \geq G\Big(\nabla h(x_k) - \mathsf{E}_k[\zeta_k], \nabla h(\phi_i^k), \frac{1}{L}(\nabla f_i(\phi_i^k) - \nabla f_i(\mathsf{x}_*))\Big). \tag{5.2}$$

Then

$$\begin{aligned} \mathsf{E}_k[D_h(x_k, z_{k+1})] \leq{}& 2L\alpha_k^2 G_k D_F(x_k, \mathsf{x}_*) + \alpha_k^2 G_k \sigma_k^2 \\ &- \frac{1}{2} D_{h^*}(\nabla h(x_k), \nabla h(x_k) - \mathsf{E}_k[\zeta_k]), \end{aligned} \tag{5.3}$$

where

$$\sigma_k^2 = 2L^2 \mathsf{E}_k\left[D_{h^*}\left(\nabla h(\phi_{i_k}^k) - \frac{1}{L}(\nabla f_{i_k}(\phi_{i_k}^k)\nabla f_{i_k}(\mathsf{x}_*)), \nabla h(\phi_{i_k}^k)\right)\right],$$

and

$$\mathsf{E}_k\left[\sigma_{k+1}^2\right] \le \left(1 - \frac{1}{n}\right)\sigma_k^2 + \frac{2L}{n} D_F(x_k, \mathsf{x}_*). \tag{5.4}$$

Remark 5.9. As we stated earlier in Remark 4.4, Lemma 5.8 shows that Assumptions 4.5 are verified with $A_k = 2LG_k$, $B_k = G_k$, $N_k = -(1/2\alpha_k^2)D_{h^*}(\nabla h(x_k), \nabla h(x_k) - \mathsf{E}_k[\zeta_k])$, $\rho = 1/n$ and $C_k = C = \frac{2L}{n}$. By just putting this values in Theorems 4.6 and 4.7 we obtain the following two corollaries, respectively.

Corollary 5.10 (F is only convex)**.** *Let assumptions of Lemma 5.8 hold. Let $(M_k)_{k\in\mathbb{N}}$ be a non-increasing positive real-valued sequence such that $M_k \ge nG_k$, $\forall k \in \mathbb{N}$. Suppose also that the sequence $(x_k)_{k\in\mathbb{N}}$ is generated by Algorithm 5.13 with $(\alpha_k)_{k\in\mathbb{N}}$ a non-decreasing positive real-valued sequence such that $\alpha_k < \frac{1}{2L(G_k+M_k/n)}$, $\forall k \in \mathbb{N}$. Then, for all $k \in \mathbb{N}$,*

$$\mathsf{E}[F(\bar{x}_k) - F(\mathsf{x}_*)] \le \frac{(1/\alpha_0^2)\mathsf{E}[D_h(\mathsf{x}_*, x_0)] + M_0\mathsf{E}[\sigma_0^2]}{\sum_{t=0}^{k-1}(1/\alpha_t)\left(1 - 2\alpha_t L(G_t + M_t/n)\right)},$$

with

$$\bar{x}_k = \sum_{t=0}^{k-1} \frac{(1/\alpha_t)\left(1 - 2\alpha_t L(G_t + M_t/n)\right)}{\sum_{t=0}^{k-1}(1/\alpha_t)\left(1 - 2\alpha_t L(G_t + M_t/n)\right)} x_t.$$

Corollary 5.11 (F is μ-relatively strongly convex)**.** *Let assumptions of Lemma 5.8 hold. Let $(M_k)_{k\in\mathbb{N}}$ be a non-increasing positive real-valued sequence such that $M_k > nG_k$, $\forall k \in \mathbb{N}$. Suppose also that the sequence $(x_k)_{k\in\mathbb{N}}$ is generated by Algorithm 5.13 with $(\alpha_k)_{k\in\mathbb{N}}$ a non-decreasing positive real-valued sequence such that $\alpha_k < \frac{1}{2L(G_k+M_k/n)}$, $\forall k \in \mathbb{N}$. Set $q_k :=$ $\max\left\{1 - \alpha_k\gamma_h\mu\left(1 - 2\alpha_k L(G_k + M_k/n)\right), 1 + \frac{G_k}{M_k} - \frac{1}{n}\right\}$. Then, for all $k \in \mathbb{N}$, $q_k \in]0, 1[$,*

$$V_{k+1} \le q_k V_k \ \ \textit{and} \ \ V_{k+1} \le \left(\prod_{t=0}^{k} q_t\right) V_0,$$

where

$$V_k = \mathsf{E}\left[\frac{1}{\alpha_k^2} D_h(\mathsf{x}_*, x_k) + M_k\sigma_k^2\right].$$

Using relative smoothness, we get

$$D_F(\mathsf{x}_*, x_{k+1}) \le \left(\prod_{t=0}^{k} q_t\right)\alpha_{k+1}^2 L V_0.$$

Remark 5.12. To see a normal linear rate, we can set $\alpha_k = \alpha = \frac{1}{2L(G_0+M_0/n)+1} \le \frac{1}{2L(G_k+M_k/n)}$, for all $k \in \mathbb{N}$. We can always take M_k such that $\frac{G_k}{M_k} = \frac{1}{n+1}$. Then $(q_k)_{k\in\mathbb{N}}$ is non-increasing,

$$V_{k+1} \le q_0 V_k \ \text{ and } \ V_{k+1} \le q_0^{k+1} V_0.$$

Also

$$D_F(\mathsf{x}_*, x_{k+1}) \le q_0^{k+1} L\mathsf{E}\left[D_h(\mathsf{x}_*, x_0) + \alpha^2 M_0\sigma_0^2\right].$$

5.3 Bregman Loopless SVRP (BLSVRP)

We also proposed BLSVRP, using an L-SVRG-style technique with only one loop. The second loop is replaced by a Bernoulli probability. It generalizes, to the Bregman distance, L-SVRP found for instance in Khaled and Jin [16] and Traoré et al. [32].

Algorithm 5.13 (**BLSVRP**)**.** Let $(i_k)_{k\in\mathbb{N}}$ be a sequence of i.i.d. random variables uniformly distributed on $\{1,\dots,n\}$ and let $(\varepsilon_k)_{k\in\mathbb{N}}$ be a sequence of i.i.d Bernoulli random variables such that $\mathsf{P}(\varepsilon_k=1)=p\in(0,1]$. Let $\alpha_k>0$ for every $k\in\mathbb{N}$, and set $x_0\equiv u_0\equiv \mathsf{x}_0\in\operatorname{int}\mathcal{C}$.

$$
\begin{array}{l}
\text{for } k=0,1,\dots\\
\left\lfloor
\begin{array}{l}
x_{k+1}=\arg\min_{\mathsf{x}\in\mathsf{H}}\left\{f_{i_k}(\mathsf{x})+\frac{1}{\alpha_k}D_h(\mathsf{x},x_k)-\langle\nabla f_{i_k}(u_k)-\nabla F(u_k),\mathsf{x}-x_k\rangle\right\}\\
u_{k+1}=(1-\varepsilon_k)u_k+\varepsilon_k x_k
\end{array}
\right.
\end{array}
$$

For BLSVRP, $e_k=\nabla f_{i_k}(u_k)-\nabla F(u_k)$. Set $\mathfrak{F}_k=\sigma(i_0,\dots,i_{k-1},\varepsilon_0,\dots,\varepsilon^{k-1})$ and $\mathsf{E}_k[\cdot]=\mathsf{E}[\cdot\,|\,\mathfrak{F}_k]$. We then have that x_k, u_k and y_k are $\mathfrak{F}_k$-measurable, i_k and ε_k are independent of $\mathfrak{F}_k$. Let $\zeta_k=-2\alpha_k(\nabla f_{i_k}(\mathsf{x}_*)-\nabla f_{i_k}(u_k))$. So, $\mathsf{E}_k[\zeta_k]=2\alpha_k\nabla F(u_k)$.

Lemma 5.14. *Suppose that Assumptions* (A.iii)*, 2.1 and* (A.v) *hold. Let $s\in\mathbb{N}$ and let $(x_k)_{k\in[m]}$ be the sequence generated by the inner iteration in Algorithm 5.13. Assume that there exists a non-increasing sequence $(G_k)_{k\in\mathbb{N}}$ such that, for all $i\in[n]$,*

$$
G_k\geq G\left(\nabla h(x_k),\nabla h(x_k),\frac{1}{L}(\nabla f_i(x_k)-\nabla f_i(\mathsf{x}_*))\right), \tag{5.5}
$$

$$
G_k\geq G\Big(\nabla h(x_k)-\mathsf{E}_k[\zeta_k],\nabla h(u_k),\frac{1}{L}(\nabla f_i(u_k)-\nabla f_i(\mathsf{x}_*))\Big). \tag{5.6}
$$

Then

$$
\begin{aligned}
\mathsf{E}_k[D_h(x_k,z_{k+1})]\leq{}& 2L\alpha_k^2G_kD_F(x_k,\mathsf{x}_*)+\alpha_k^2G_k\sigma_k^2\\
&-\frac{1}{2}D_{h^*}(\nabla h(x_k),\nabla h(x_k)-\mathsf{E}_k[\zeta_k]),
\end{aligned}\tag{5.7}
$$

where

$$
\sigma_k^2=2L^2\mathsf{E}_k\left[D_{h^*}\Big(\nabla h(u_k)-\frac{1}{L}(\nabla f_{i_k}(u_k)-\nabla f_{i_k}(\mathsf{x}_*)),\nabla h(u_k)\Big)\right],
$$

and

$$
\mathsf{E}_k[\sigma_{k+1}^2]\leq(1-p)\sigma_k^2+2pLD_F(x_k,\mathsf{x}_*).
$$

Remark 5.15. Lemma 5.14 shows that Assumptions 4.5 are verified with $A_k=2LG_k$, $B_k=G_k$, $N_k=-(1/2\alpha_k^2)D_{h^*}(\nabla h(x_k),\nabla h(x_k)-\mathsf{E}_k[\zeta_k])$, $\rho=p$ and $C_k=C=2pL$.

Corollary 5.16 (F is only convex)**.** *Let assumptions of Lemma 5.14 hold. Let $(M_k)_{k\in\mathbb{N}}$ be a non-increasing positive real-valued sequence such that $M_k\geq\frac{G_k}{p}$, $\forall k\in\mathbb{N}$. Suppose also that the sequence $(x_k)_{k\in\mathbb{N}}$ is generated by Algorithm 5.13 with $(\alpha_k)_{k\in\mathbb{N}}$ a non-decreasing positive real-valued sequence such that $\alpha_k<\frac{1}{2L(G_k+M_kp)}$, $\forall k\in\mathbb{N}$. Then, for all $k\in\mathbb{N}$,*

$$
\mathsf{E}[F(\bar{x}_k)-F(\mathsf{x}_*)]\leq\frac{(1/\alpha_0^2)\mathsf{E}[D_h(\mathsf{x}_*,x_0)]+M_0\mathsf{E}[\sigma_0^2]}{\sum_{t=0}^{k-1}(1/\alpha_t)\left(1-2\alpha_tL(G_t+M_tp)\right)},
$$

with

$$
\bar{x}_k=\sum_{t=0}^{k-1}\frac{(1/\alpha_t)\left(1-2\alpha_tL(G_t+M_tp)\right)}{\sum_{t=0}^{k-1}(1/\alpha_t)\left(1-2\alpha_tL(G_t+M_tp)\right)}x_t.
$$

Corollary 5.17 (F is μ-relatively strongly convex)**.** *Let assumptions of Lemma 5.14 hold. Let* $(M_k)_{k\in\mathbb{N}}$ *be a non-increasing positive real-valued sequence such that* $M_k > \frac{G_k}{p}$, $\forall k \in \mathbb{N}$. *Suppose also that the sequence* $(x_k)_{k\in\mathbb{N}}$ *is generated by Algorithm 5.13 with* $(\alpha_k)_{k\in\mathbb{N}}$ *a non-decreasing positive real-valued sequence such that* $\alpha_k < \frac{1}{2L(G_k+M_kp)}$, $\forall k \in \mathbb{N}$. *Set* $q_k \coloneqq \max\left\{1 - \alpha_k\gamma_h\mu\left(1 - 2\alpha_k L(G_k + M_k p)\right), 1 + \frac{G_k}{M_k} - p\right\}$. *Then, for all* $k \in \mathbb{N}$, $q_k \in]0, 1[$,

$$V_{k+1} \leq q_k V_k \ \ \textit{and} \ \ V_{k+1} \leq \left(\prod_{t=0}^{k} q_t\right) V_0,$$

where

$$V_k = \mathsf{E}\left[\frac{1}{\alpha_k^2} D_h(\mathsf{x}_*, x_k) + M_k\sigma_k^2\right].$$

Using relative smoothness, we get

$$D_F(\mathsf{x}_*, x_{k+1}) \leq \left(\prod_{t=0}^{k} q_t\right) \alpha_{k+1}^2 L V_0.$$

6 Experiments

We illustrate the advantage of variance reduction for BSAPA by performing in this section a numerical experiment on the Poisson linear inverse problem. It is given by:

$$\min_{\mathsf{x}\in\mathbb{R}_+^d} F(\mathsf{x}) = \frac{1}{n} D_{\mathrm{KL}}(b, A\mathsf{x}),$$

where

$$D_{\mathrm{KL}}(b, A\mathsf{x}) \coloneqq \sum_{i=1}^{n} \left\{ f_i(\mathsf{x}) \coloneqq \mathsf{b}_i \log(\mathsf{b}_i/(A\mathsf{x})_i) - \mathsf{b}_i + (A\mathsf{x})_i \right\}$$

is the Kullback-Leibler divergence, $A \in \mathbb{R}_+^{n\times d}$ is the forward operator of the inverse problem, and $b \in \mathbb{R}_{++}^n$ is the measurements vector. It models the maximum likelihood estimation problem where the model is $b \sim \mathrm{Poisson}(A\mathsf{x}_*)$, with x_* the true unknown value. Each f_i is convex. The kernel used is $h(\mathsf{x}) = -\sum_{i=0}^{d} \log \mathsf{x}_i$. Based on Bauschke et al. [3], we have that each function f_i is L-relatively smooth w.r.t. h when $L = \max_i b_i$. In these experiments, we compare a fixed stepsize BSAPA to non variance-reduced BSPPA with both fixed and vanishing stepsizes.

At each iteration, we solved the proximal subproblem

$$\operatorname*{arg\,min}_{\mathsf{x}\in\mathbb{R}_+^d} \left\{ f_{i_k}(\mathsf{x}) - \langle e_k, \mathsf{x} - x_k\rangle + \tfrac{1}{\alpha_k} D_h(\mathsf{x}, x_k) \right\}$$

using a gradient descent subroutine, since no closed-form solution is available in general. As this subroutine is employed in both algorithms under comparison, the evaluation remains equitable. Strictly speaking, this procedure corresponds to an approximate minimization of the proximal mapping, and hence the implementation is closer to an inexact variant of the algorithms. A formal treatment of inexactness lies beyond the scope of this work. Nevertheless, this approximation does not affect our empirical findings, which still clearly demonstrate the advantage of variance reduction. These findings encourage future extensions of the theoretical analysis toward inexact variants.

For the first experiment, we used a uniformly generated matrix. We set $n = 500, d = 100$. We couldn't choose the sequence $(G_k)_{k\in\mathbb{N}}$, as the existence of a gain function for the kernel is unknown. Following previous work [11] and their numerical experiments, we therefore assume $(G_k)_{k\in\mathbb{N}}$ is constant, which can possibly go beyond our theoretical analysis. To account for this uncertainty and also the fact that L is quite conservative, we consider several values for the stepsize in our experiments. Here, we present the results for two values; see Figure 1. For the complete range of stepsizes, see Appendix C, where all additional experimental results are provided. As expected, the results show that with vanishing stepsize (green curve), SPPA is slower that BSAPA (blue curve), and even plateaus quite early. With constant stepsize (orange curve), SPPA has the same rate as BSAPA (interpolation). However, while the stepsizes increase we observe that SPPA is less stable than BSAPA. More stability can be added to SPPA by using average iterates $(\bar{x})_{k\in\mathbb{N}}$. However this instability of SPPA, in the interpolation case, is vastly due to the approximate proximal mapping. Indeed when the A is diagonal, a closed form exists for the proximal mapping and the experiments don't exhibit this instability for SPPA; see Figure 2. Nevertheless, the experiments shows the advantage and stability of variance reduction, even in the interpolation case, when the proximal mapping can only be approximated.

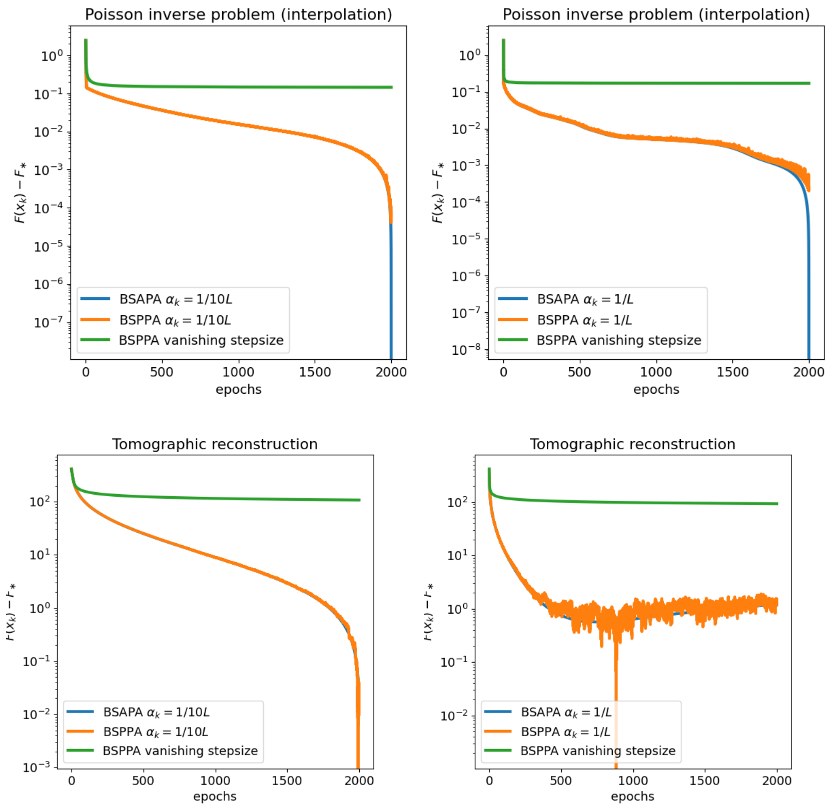

Figure 1: Our BSAPA (variance reduced) is more stable, converges to the minimum and does not oscillate around it, even in the non-interpolation case with constant stepsize, contrary to BSPPA.

To cover the non-interpolation case, we did some experiments on tomographic reconstruction problem, where A is a discrete Radon transform, that projects the image x in n different angles $(\theta_1, \ldots, \theta_n)$, with $n = 90$. The optimization problem is

$$\underset{\mathsf{x}\in\mathbb{R}^d_+}{\text{minimize}} \ \frac{1}{n}\sum_{i=1}^{n}\left\{f_i(\mathsf{x}) := \mathsf{b}_i \log(\mathsf{b}_i/(A\mathsf{x})_{\theta_i}) - \mathsf{b}_i + (A\mathsf{x})_{\theta_i}\right\}.$$

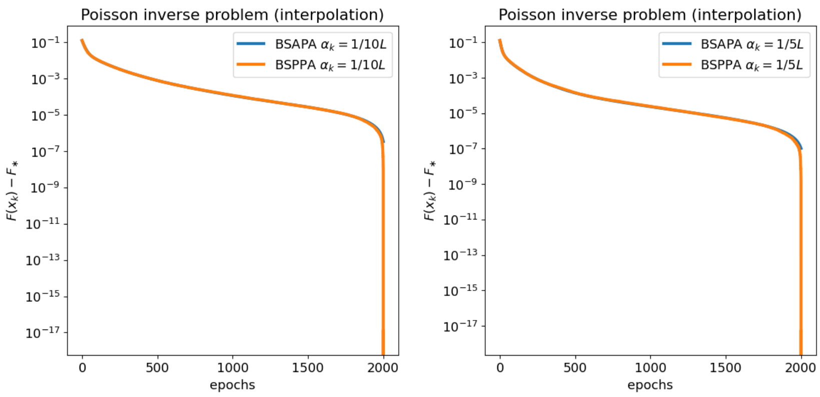

Figure 2: The case of diagonal A: closed-form solution to the proximal mapping. There is no instability for BSPPA in that case, showing that its instability in Figure 1 is due to the inexact proximal mapping. This proves it is less stable to inexact algorithms compare to its variance-reduced counterparts.

The image used is the Shepp-Logan phantom. Also here, the vanishing stepsize SPPA is slower than BSAPA; see Figure 1. Furthermore, the results show, that constant stepsize SPPA oscillates around the minimum while BSAPA is stable; expected behaviors for non-interpolation cases. These oscillations are low for small stepsizes and high for bigger ones. We also ran this for different stepsizes because G_k is unknown and L is conservative.

Finally, in all experiments, BSAPA tends to explode quicker than SPPA when the stepsize is big enough. This is seen from the theory where SPPA does not have a bound on the stepsize; see Theorems 5.2 and 5.4. However bigger stepsizes increase the term which is related to the variance in those theorems, hence an increase in oscillations and no convergence. But until that stepsize threshold imposed by the theory, for non-interpolation cases, BSAPA remains more stable and ensure convergence whereas SPPA does not.

7 Conclusion

In this paper, we conducted a unified analysis of variance reduction for the Bregman stochastic proximal point algorithm (BSPPA). We proposed a generic variance-reduced algorithm based on BSPPA and prove convergence rates. From that general algorithm and analysis, we derived several new variance-reduced BSPPAs, employing SVRG- and SAGA-like techniques, and their corresponding convergence rates. More specifically, under the relative smoothness assumption, we prove sublinear and linear rates for convex and relatively strongly convex functions, respectively. Our general analysis can also recover the previously studied vanilla BSPPA with its standard rates. Furthermore, the unified theoretical results extend seamlessly to variance-reduced SGD with Bregman distance. For future work, we will focus on extending this work to nonsmooth functions and/or to inexact proximal mapping variants.

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Appendices

A Bregman SVRP (BSVRP)

Our analysis can also be adapted to the original SVRG-like variance reduction, i.e. a Bregman version of SVRP in Traoré et al. [32]. We call it BSVRP.

Algorithm A.1 (**BSVRP).** Let $m \in \mathbb{N}$, with $m \geq 1$, and $(\xi_s)_{s\in\mathbb{N}}$, $(i_t)_{t\in\mathbb{N}}$ be two independent sequences of i.i.d. random variables uniformly distributed on $\{0, 1, \dots, m-1\}$ and $\{1, \dots, n\}$ respectively. Let $\alpha_k > 0$ for every $k \in \mathbb{N}$, and set $\tilde{x}_0 \equiv \tilde{\mathsf{x}}_0 \in \operatorname{int} \mathcal{C}$.

$$
\begin{array}{l}
\text{for } s = 0, 1, \dots \\
\left\lfloor
\begin{array}{l}
x_0 = \tilde{x}_s \\
\text{for } k = 0, \dots, m-1 \\
\left\lfloor
\begin{array}{l}
x_{k+1} = \arg\min_{\mathsf{x}\in\mathsf{H}} \left\{ f_{i_k}(\mathsf{x}) + \frac{1}{\alpha_k} D_h(\mathsf{x}, x_k) - \langle \nabla f_{i_{sm+k}}(\tilde{x}_s) - \nabla F(\tilde{x}_s), \mathsf{x} - x_k \rangle \right\}
\end{array}
\right. \\
\tilde{x}_{s+1} = \sum_{k=0}^{m-1} \delta_{k,\xi_s} x_k, \ \text{ or } \tilde{x}_{s+1} = \frac{1}{m} \sum_{k=0}^{m-1} x_k,
\end{array}
\right.
\end{array}
$$

where $\delta_{k,h}$ is the Kronecker symbol. It follows that $e_k = \nabla f_{i_{sm+k}}(\tilde{x}_s) - \nabla F(\tilde{x}_s)$. Moreover, setting $\mathfrak{F}_{s,k} = \sigma(\xi_0, \dots, \xi_{s-1}, i_0, \dots, i_{sm+k-1})$, we have that $\tilde{x}_s, \mathsf{x}_*$, and x_k are $\mathfrak{F}_{s,k}$-measurable and i_{sm+k} is independent of $\mathfrak{F}_{s,k}$. Let $\zeta_k = -2\alpha_k(\nabla f_{i_k}(\mathsf{x}_*) - \nabla f_{i_k}(\tilde{x}_s))$. That means that $\mathsf{E}[\zeta_k \,|\, \mathfrak{F}_{s,k}] = 2\alpha_k \nabla F(\tilde{x}_s)$.

Lemma A.2. *Suppose that Assumptions* (A.iii)*, 2.1 and* (A.v) *hold. Let $s \in \mathbb{N}$ and let $(x_k)_{k\in[m]}$ be the sequence generated by the inner iteration in Algorithm A.1. We assume that there exists a non-increasing sequence $(G_k)_{k\in\mathbb{N}}$ such that, for all $i \in [n]$,*

$$G_k \geq G\left(\nabla h(x_k), \nabla h(x_k), \frac{1}{L}(\nabla f_i(x_k) - \nabla f_i(\mathsf{x}_*))\right), \tag{A.1}$$

$$G_k \geq G\Big(\nabla h(x_k) - \mathsf{E}[\zeta_k \,|\, \mathfrak{F}_{s,k}], \nabla h(\tilde{x}_s), \frac{1}{L}(\nabla f_i(\tilde{x}_s) - \nabla f_i(\mathsf{x}_*))\Big). \tag{A.2}$$

Then, for all $k \in \{0, 1, \cdots, m-1\}$,

$$\mathsf{E}[D_h(x_k, z_{k+1}) \,|\, \mathfrak{F}_{s,k}] \leq 2L\alpha_k^2 G_k D_F(x_k, \mathsf{x}_*) + \alpha_k^2 G_k \sigma_k^2 - \frac{1}{2} D_{h^*}\left(\nabla h(x_k), \nabla h(x_k) + 2\alpha_k \nabla F(\tilde{x}_s)\right),$$

where

$$\sigma_k^2 = 2L^2 \mathsf{E}\left[D_{h^*}\Big(\nabla h(\tilde{x}_s) - \frac{1}{L}(\nabla f_{i_k}(\tilde{x}_s) - \nabla f_{i_k}(\mathsf{x}_*)), \nabla h(\tilde{x}_s)\Big) \,|\, \mathfrak{F}_{s,k}\right],$$

and, trivially, $\mathsf{E}[\sigma_{k+1}^2 \,|\, \mathfrak{F}_{s,k}] = \mathsf{E}[\sigma_k^2 \,|\, \mathfrak{F}_{s,k}]$.

Remark A.3. By Lemma A.2, Assumptions 4.5 are satisfied with $A_k = 2LG_k$, $B_k = G_k$, $N_k = -(1/2\alpha_k^2) D_{h^*}\left(\nabla h(x_k), \nabla h(x_k) + 2\alpha_k \nabla F(\tilde{x}_s)\right)$, $\rho = C_k = 0$. For simplicity, for any $k \in \mathbb{N}$, we set $A_k = A = 2LG_k = 2LG_0$, $B_k = B = G_k = G_0$, $\alpha_k = \alpha$.

Remark A.4. For simplicity, for any $k \in \mathbb{N}$, we set $A_k = A = 2LG_k = 2LG_0$, $B_k = B = G_k = G_0$, $\alpha_k = \alpha$.

Theorem A.5 (F is μ-relatively strongly convex)**.** *We assume* (A.iv) *and that $\alpha < \frac{1}{4LG_0}$. Then, for all $s \in \mathbb{N}$ and under the assumptions of Lemma A.2,*

$$\mathsf{E}[D_F(\tilde{x}_{s+1}, \mathsf{x}_*)] \leq \left(\frac{1}{\gamma_h \mu \alpha\,(1 - 2L\alpha G_0)\, m} + \frac{2L\alpha G_0}{1 - 2L\alpha G_0}\right) \mathsf{E}[D_F(\tilde{x}_s, \mathsf{x}_*)].$$

Remark A.6. Taking $\alpha < \dfrac{1}{4LG_0}$ ensures that $\dfrac{2L\alpha G_0}{1-2L\alpha G_0} < 1$. Then taking m big enough gives linear rate.

B Proofs

B.1 Fundamental results

For the proofs, we start with a proposition that constitutes the cornerstone of our analysis. Most of the others results are derived from that proposition.

Proposition B.1. *Suppose that Assumptions 4.5 are verified, Assumption* (A.ii) *holds, and that the sequence* $(x_k)_{k\in\mathbb{N}}$ *is generated by Algorithm 4.1. Then, for all* $k \in \mathbb{N}$,

$$(1+\beta\alpha_k)\,\mathsf{E}[D_h(\mathsf{x}_*, x_{k+1})\,|\,\mathfrak{F}_k] \leq D_h(\mathsf{x}_*, x_k) - \alpha_k\left[1-\alpha_k A_k\right](F(x_k)-F(\mathsf{x}_*)) + \alpha_k^2 B_k \sigma_k^2 + \alpha_k^2 N_k.$$

Proof. We recall the definition of x_{k+1}:

$$\begin{aligned} x_{k+1} &= \underset{\mathsf{x}\in\mathbb{R}^d}{\arg\min}\Big\{\underbrace{f_{i_k}(\mathsf{x}) - \langle e_k, \mathsf{x}-x_k\rangle - \beta D_h(\mathsf{x}, x_k)}_{:=R_{i_k}(\mathsf{x})} + \left(\frac{1+\beta\alpha_k}{\alpha_k}\right) D_h(\mathsf{x}, x_k)\Big\} \\ &= \nabla h^*\left(\nabla h(x_k) - \left(\frac{1+\beta\alpha_k}{\alpha_k}\right)^{-1} r_{k+1}\right), \end{aligned} \tag{B.1}$$

where $r_{k+1} \in \partial R(x_{k+1})$ such that (B.1) holds.
Let $\mathsf{x} \in \mathcal{C}$. The function $R_{i_k}(\mathsf{x}) = f_{i_k}(\mathsf{x}) - \langle e_k, \mathsf{x}-x_k\rangle - \beta D_h(\mathsf{x}, x_k)$ is convex, since f_{i_k} is β-relatively strongly convex w.r.t. h. By convexity of R_{i_k}, we have:

$$\begin{aligned} \langle r_{k+1}, \mathsf{x}-x_{k+1}\rangle &\leq R_{i_k}(\mathsf{x}) - R_{i_k}(x_{k+1}) \\ &= f_{i_k}(\mathsf{x}) - f_{i_k}(x_{k+1}) - \langle e_k, \mathsf{x}-x_k\rangle + \langle e_k, x_{k+1}-x_k\rangle \\ &\quad - \beta D_h(\mathsf{x}, x_k) + \beta D_h(x_{k+1}, x_k). \end{aligned} \tag{B.2}$$

Since $r_{k+1} = \left(\frac{1+\beta\alpha_k}{\alpha_k}\right)(\nabla h(x_k) - \nabla h(x_{k+1}))$ by (B.1), it follows from (B.2):

$$\begin{aligned} &\left(\frac{1+\beta\alpha_k}{\alpha_k}\right)\langle \nabla h(x_k) - \nabla h(x_{k+1}), \mathsf{x}-x_{k+1}\rangle \\ &\leq f_{i_k}(\mathsf{x}) - f_{i_k}(x_{k+1}) - \langle e_k, \mathsf{x}-x_k\rangle + \langle e_k, x_{k+1}-x_k\rangle \\ &\quad - \beta D_h(\mathsf{x}, x_k) + \beta D_h(x_{k+1}, x_k). \end{aligned}$$

By using the three points identity in Lemma 2.5, $\langle \nabla h(x_k) - \nabla h(x_{k+1}), \mathsf{x}-x_{k+1}\rangle = D_h(x_{k+1}, x_k) + D_h(\mathsf{x}, x_{k+1}) - D_h(\mathsf{x}, x_k)$, in the previous inequality, we find

$$\begin{aligned} -\langle e_k, x_{k+1}-x_k\rangle + f_{i_k}(x_{k+1}) - f_{i_k}(\mathsf{x}) + \frac{1}{\alpha_k} D_h(x_{k+1}, x_k) &\leq \frac{1}{\alpha_k} D_h(\mathsf{x}, x_k) - \frac{(1+\beta\alpha_k)}{\alpha_k} D_h(\mathsf{x}, x_{k+1}) \\ &\quad - \langle e_k, \mathsf{x}-x_k\rangle. \end{aligned} \tag{B.3}$$

We recall that the explicit direction v_k can be rewritten as

$$v_k = \frac{1}{\alpha_k}\left(\nabla h(x_k) - \nabla h(z_{k+1})\right). \tag{B.4}$$

Using the definitions of v_k in (B.4) and (4.3), we can lower bound the left hand side as follows:

$$\begin{aligned}
-\langle e_k, x_{k+1}-x_k\rangle + f_{i_k}(x_{k+1}) - f_{i_k}(\mathsf{x}) + \frac{1}{\alpha_k}D_h(x_{k+1},x_k) &= -\langle e_k, x_{k+1}-x_k\rangle + f_{i_k}(x_{k+1}) - f_{i_k}(x_k) \\
&\quad + \frac{1}{\alpha_k}D_h(x_{k+1},x_k) + f_{i_k}(x_k) - f_{i_k}(\mathsf{x}) \\
&\geq -\langle e_k, x_{k+1}-x_k\rangle + \langle g_k, x_{k+1}-x_k\rangle \\
&\quad + \frac{1}{\alpha_k}D_h(x_{k+1},x_k) + f_{i_k}(x_k) - f_{i_k}(\mathsf{x}) \\
&= \langle -e_k + g_k, x_{k+1}-x_k\rangle + \frac{1}{\alpha_k}D_h(x_{k+1},x_k) \\
&\quad + f_{i_k}(x_k) - f_{i_k}(\mathsf{x}) \\
&= \langle v_k, x_{k+1}-x_k\rangle + \frac{1}{\alpha_k}D_h(x_{k+1},x_k) \\
&\quad + f_{i_k}(x_k) - f_{i_k}(\mathsf{x}) \\
&= \frac{1}{\alpha_k}\langle \nabla h(x_k) - \nabla h(z_{k+1}), x_{k+1}-x_k\rangle \\
&\quad + \frac{1}{\alpha_k}D_h(x_{k+1},x_k) + f_{i_k}(x_k) - f_{i_k}(\mathsf{x}) \\
&= -\frac{1}{\alpha_k}D_h(x_k,z_{k+1}) + \frac{1}{\alpha_k}D_h(x_{k+1},z_{k+1}) \\
&\quad + f_{i_k}(x_k) - f_{i_k}(\mathsf{x}), \qquad \text{(B.5)}
\end{aligned}$$

where in the first inequality, we used convexity of f_i, for all $i \in [n]$ and in the last one uses the three points inequality (2.5). By using (B.5) in (B.3) and $D_h(x_{k+1}, z_{k+1}) \geq 0$, we obtain

$$\begin{aligned}
-\frac{1}{\alpha_k}D_h(x_k,z_{k+1}) + f_{i_k}(x_k) - f_{i_k}(\mathsf{x}) \leq \frac{1}{\alpha_k}D_h(\mathsf{x},x_k) &- \frac{(1+\beta\alpha_k)}{\alpha_k}D_h(\mathsf{x},x_{k+1}) \\
&- \langle e_k, \mathsf{x}-x_k\rangle. \qquad \text{(B.6)}
\end{aligned}$$

Now, define $\mathsf{E}_k[\cdot] = \mathsf{E}[\cdot \,|\, \mathfrak{F}_k]$, where $\mathfrak{F}_k$ is defined in Assumptions 4.5 and is such that x_k is $\mathfrak{F}_k$-measurable and i_k is independent of $\mathfrak{F}_k$. Thus, taking the conditional expectation in (B.6) and rearranging the terms, we have

$$\begin{aligned}
(1+\beta\alpha_k)\,\mathsf{E}_k[D_h(\mathsf{x},x_{k+1})] &\leq D_h(\mathsf{x},x_k) - \alpha_k\mathsf{E}_k\big[f_{i_k}(x_k) - f_{i_k}(\mathsf{x})\big] + \mathsf{E}_k[D_h(x_k,z_{k+1})] \\
&= D_h(\mathsf{x},x_k) - \alpha_k(F(x_k) - F(\mathsf{x})) + \mathsf{E}_k[D_h(x_k,z_{k+1})].
\end{aligned}$$

Replacing x by x_* and using Assumption (B.ii), we get

$$\begin{aligned}
(1+\beta\alpha_k)\,\mathsf{E}_k[D_h(\mathsf{x}_*,x_{k+1})] &\leq D_h(\mathsf{x}_*,x_k) - \alpha_k(F(x_k) - F(\mathsf{x}_*)) \\
&\quad + \Big(\alpha_k^2 A_k\,(F(x_k) - F(\mathsf{x}_*)) + \alpha_k^2 B_k\sigma_k^2 + \alpha_k^2 N_k\Big) \\
&= D_h(\mathsf{x}_*,x_k) + \alpha_k^2 N_k \\
&\quad - \alpha_k(1-\alpha_k A_k)(F(x_k) - F(\mathsf{x}_*)) + \alpha_k^2 B_k\sigma_k^2. \qquad \square
\end{aligned}$$

Remark B.2. We show in the next proposition that the same result in Proposition B.1 stands for Bregman SGD with the generic variance reduction term e_k. As a consequence, since all the variance reduced results of the paper stem from Proposition B.1, they are also true for Bregman SGD with the same variance reduction techniques.

Proposition B.3 (SGD case)**.** *Suppose that Assumptions 4.5 are verified, Assumption* (A.ii) *holds, and that the sequence* $(x_k)_{k\in\mathbb{N}}$ *is generated by the explicit version of Algorithm 4.1,i.e., SGD with the generic variance reduction term* (4.4)*. Then, for all* $k \in \mathbb{N}$*,*

$$\begin{aligned}(1+\beta\alpha_k)\,\mathsf{E}[D_h(\mathsf{x}_*, x_{k+1}) \,|\, \mathfrak{F}_k] &\coloneqq (1+\beta\alpha_k)\,\mathsf{E}[D_h(\mathsf{x}_*, z_{k+1}) \,|\, \mathfrak{F}_k] \\ &\leq D_h(\mathsf{x}_*, x_k) - \alpha_k\left[1-\alpha_k A_k\right](F(x_k) - F(\mathsf{x}_*)) + \alpha_k^2 B_k \sigma_k^2 + \alpha_k^2 N_k.\end{aligned}$$

Proof. We recall the definition of $x_{k+1} \coloneqq z_{k+1}$:

$$\begin{aligned} x_{k+1} &= \operatorname*{arg\,min}_{\mathsf{x}\in\mathbb{R}^d} \Big\{ \underbrace{\langle g_k, \mathsf{x}-x_k\rangle - \langle e_k, \mathsf{x}-x_k\rangle - \beta D_h(\mathsf{x}, x_k)}_{:=R_{i_k}(\mathsf{x})} + \left(\frac{1+\beta\alpha_k}{\alpha_k}\right) D_h(\mathsf{x}, x_k) \Big\} \\ &= \nabla h^*\left(\nabla h(x_k) - \left(\frac{1+\beta\alpha_k}{\alpha_k}\right)^{-1} r_{k+1}\right), \end{aligned} \tag{B.7}$$

where $g_k \in \partial f_{i_k}(x_k)$ and $r_{k+1} \in \partial R(x_{k+1})$ such that (B.7) holds.
Let $\mathsf{x} \in \mathcal{C}$. The function $R_{i_k}(\mathsf{x}) = \langle g_k, \mathsf{x}-x_k\rangle - \langle e_k, \mathsf{x}-x_k\rangle - \beta D_h(\mathsf{x}, x_k)$ is convex, since f_{i_k} is β-relatively strongly convex w.r.t. h. By convexity of R_{i_k}, we have:

$$\begin{aligned}\langle r_{k+1}, \mathsf{x}-x_{k+1}\rangle &\leq R_{i_k}(\mathsf{x}) - R_{i_k}(x_{k+1}) \\ &= \langle g_k, \mathsf{x}-x_k\rangle - \langle g_k, x_{k+1}-x_k\rangle - \langle e_k, \mathsf{x}-x_k\rangle + \langle e_k, x_{k+1}-x_k\rangle \\ &\quad - \beta D_h(\mathsf{x}, x_k) + \beta D_h(x_{k+1}, x_k). \end{aligned}\tag{B.8}$$

Since $r_{k+1} = \left(\frac{1+\beta\alpha_k}{\alpha_k}\right)(\nabla h(x_k) - \nabla h(x_{k+1}))$ by (B.7), it follows from (B.8) and the convexity of f_{i_k}:

$$\begin{aligned}&\left(\frac{1+\beta\alpha_k}{\alpha_k}\right)\langle \nabla h(x_k) - \nabla h(x_{k+1}), \mathsf{x}-x_{k+1}\rangle \\ &\leq f_{i_k}(\mathsf{x}) - f_{i_k}(x_k) - \langle g_k, x_{k+1}-x_k\rangle - \langle e_k, \mathsf{x}-x_k\rangle + \langle e_k, x_{k+1}-x_k\rangle \\ &\quad - \beta D_h(\mathsf{x}, x_k) + \beta D_h(x_{k+1}, x_k) \\ &= f_{i_k}(\mathsf{x}) - f_{i_k}(x_k) - \langle v_k, x_{k+1}-x_k\rangle - \langle e_k, \mathsf{x}-x_k\rangle \\ &\quad - \beta D_h(\mathsf{x}, x_k) + \beta D_h(x_{k+1}, x_k). \end{aligned}\tag{B.9}$$

By using the three points identity in Lemma 2.5, $\langle \nabla h(x_k) - \nabla h(x_{k+1}), \mathsf{x}-x_{k+1}\rangle = D_h(x_{k+1}, x_k) + D_h(\mathsf{x}, x_{k+1}) - D_h(\mathsf{x}, x_k)$, in the (B.9), we find

$$\begin{aligned}\langle v_k, x_{k+1}-x_k\rangle + f_{i_k}(x_k) - f_{i_k}(\mathsf{x}) + \frac{1}{\alpha_k} D_h(x_{k+1}, x_k) &\leq \frac{1}{\alpha_k} D_h(\mathsf{x}, x_k) - \frac{(1+\beta\alpha_k)}{\alpha_k} D_h(\mathsf{x}, x_{k+1}) \\ &\quad - \langle e_k, \mathsf{x}-x_k\rangle. \end{aligned}\tag{B.10}$$

We recall that the explicit direction v_k can be rewritten as

$$v_k = \frac{1}{\alpha_k}\left(\nabla h(x_k) - \nabla h(z_{k+1})\right) = \frac{1}{\alpha_k}\left(\nabla h(x_k) - \nabla h(x_{k+1})\right). \tag{B.11}$$

Using the definitions of v_k in (B.11), we can express the left hand side as follows:

$$\begin{aligned}\langle v_k, x_{k+1}-x_k\rangle + f_{i_k}(x_k) - f_{i_k}(\mathsf{x}) + \frac{1}{\alpha_k} D_h(x_{k+1}, x_k) &= \frac{1}{\alpha_k}\langle \nabla h(x_k) - \nabla h(x_{k+1}), x_{k+1}-x_k\rangle \\ &\quad + \frac{1}{\alpha_k} D_h(x_{k+1}, x_k) + f_{i_k}(x_k) - f_{i_k}(\mathsf{x}) \\ &= -\frac{1}{\alpha_k} D_h(x_k, x_{k+1}) + f_{i_k}(x_k) - f_{i_k}(\mathsf{x}), \end{aligned}\tag{B.12}$$

where in the second equality, we used convexity the three points inequality (2.5). By using (B.12) in (B.10), we obtain

$$-\frac{1}{\alpha_k}D_h(x_k,x_{k+1})+f_{i_k}(x_k)-f_{i_k}(\mathsf{x})\le\frac{1}{\alpha_k}D_h(\mathsf{x},x_k)-\frac{(1+\beta\alpha_k)}{\alpha_k}D_h(\mathsf{x},x_{k+1})\\ -\langle e_k,\mathsf{x}-x_k\rangle. \tag{B.13}$$

Now, define $\mathsf{E}_k[\cdot]=\mathsf{E}[\cdot\,|\,\mathfrak{F}_k]$, where $\mathfrak{F}_k$ is defined in Assumptions 4.5 and is such that x_k is $\mathfrak{F}_k$-measurable and i_k is independent of $\mathfrak{F}_k$. Thus, taking the conditional expectation in (B.13) and rearranging the terms, we have

$$\begin{aligned}(1+\beta\alpha_k)\,\mathsf{E}_k[D_h(\mathsf{x},x_{k+1})]&\le D_h(\mathsf{x},x_k)-\alpha_k\mathsf{E}_k\left[f_{i_k}(x_k)-f_{i_k}(\mathsf{x})\right]+\mathsf{E}_k[D_h(x_k,x_{k+1})]\\&=D_h(\mathsf{x},x_k)-\alpha_k(F(x_k)-F(\mathsf{x}))+\mathsf{E}_k[D_h(x_k,x_{k+1})].\end{aligned}$$

Replacing x by x_* and using Assumption (B.ii), we get

$$\begin{aligned}(1+\beta\alpha_k)\,\mathsf{E}_k[D_h(\mathsf{x}_*,x_{k+1})]&\le D_h(\mathsf{x}_*,x_k)-\alpha_k(F(x_k)-F(\mathsf{x}_*))\\&\qquad+\left(\alpha_k^2A_k\,(F(x_k)-F(\mathsf{x}_*))+\alpha_k^2B_k\sigma_k^2+\alpha_k^2N_k\right)\\&=D_h(\mathsf{x}_*,x_k)+\alpha_k^2N_k\\&\qquad-\alpha_k(1-\alpha_kA_k)(F(x_k)-F(\mathsf{x}_*))+\alpha_k^2B_k\sigma_k^2.\end{aligned}$$ □

B.2 Technical lemmas

These following technical lemmas are needed in the proofs.

Lemma B.4. *Suppose that Assumptions 4.5 are verified and that the sequence $(x_k)_{k\in\mathbb{N}}$ is generated by Algorithm 4.1 with $(\alpha_k)_{k\in\mathbb{N}}$ a non-decreasing positive real-valued sequence. Let $(M_k)_{k\in\mathbb{N}}$ be a non-increasing positive real-valued sequence. Then, for all $k\in\mathbb{N}$,*

$$\begin{aligned}\frac{1}{\alpha_{k+1}^2}\mathsf{E}[D_h(\mathsf{x}_*,x_{k+1})]+M_{k+1}\mathsf{E}[\sigma_{k+1}^2]&\le\frac{1}{\alpha_k^2}\mathsf{E}[D_h(\mathsf{x}_*,x_k)]+(M_k+B_k-\rho M_k)\mathsf{E}[\sigma_k^2]\\&\quad-\frac{1}{\alpha_k}\,(1-\alpha_k(A_k+M_kC_k))\,\mathsf{E}[F(x_k)-F(\mathsf{x}_*)]\\&\quad+\mathsf{E}[N_k].\end{aligned}$$

Proof. Define $\mathsf{E}_k[\cdot]=\mathsf{E}[\cdot\,|\,\mathfrak{F}_k]$. It follows from Proposition B.1 that

$$\begin{aligned}\frac{1}{\alpha_{k+1}^2}\mathsf{E}_k[D_h(\mathsf{x}_*,x_{k+1})]+M_{k+1}\mathsf{E}_k[\sigma_{k+1}^2]&\le\frac{1}{\alpha_k^2}\mathsf{E}_k[D_h(\mathsf{x}_*,x_{k+1})]+M_k\mathsf{E}_k[\sigma_{k+1}^2]\\&\le\frac{1}{\alpha_k^2}D_h(\mathsf{x}_*,x_k)+N_k+B_k\sigma_k^2+M_k\mathsf{E}_k[\sigma_{k+1}^2]\\&\qquad-\frac{1}{\alpha_k}(1-\alpha_kA_k)(F(x_k)-F(\mathsf{x}_*)).\end{aligned}$$

By taking the total expectation, we get

$$\begin{aligned}
\frac{1}{\alpha_{k+1}^2}\mathsf{E}[D_h(\mathsf{x}_*, x_{k+1})] + M_{k+1}\mathsf{E}[\sigma_{k+1}^2] &\le \frac{1}{\alpha_k^2}\mathsf{E}[D_h(\mathsf{x}_*, x_k)] + \mathsf{E}[N_k] + B_k\mathsf{E}[\sigma_k^2] + M_k\mathsf{E}[\sigma_{k+1}^2] \\
&\quad - \frac{1}{\alpha_k}(1-\alpha_k A_k)\mathsf{E}[F(x_k) - F(\mathsf{x}_*)] \\
&\le \frac{1}{\alpha_k^2}\mathsf{E}[D_h(\mathsf{x}_*, x_k)] + \mathsf{E}[N_k] + B_k\mathsf{E}[\sigma_k^2] \\
&\quad - \frac{1}{\alpha_k}(1-\alpha_k A_k)\mathsf{E}[F(x_k) - F(\mathsf{x}_*)] \\
&\quad + M_k(1-\rho)\mathsf{E}\left[\sigma_k^2\right] + M_k C_k \mathsf{E}[F(x_k) - F(\mathsf{x}_*)] \\
&= \frac{1}{\alpha_k^2}\mathsf{E}[D_h(\mathsf{x}_*, x_k)] + \mathsf{E}[N_k] + (M_k + B_k - \rho M)\mathsf{E}[\sigma_k^2] \\
&\quad - \frac{1}{\alpha_k}\left(1 - \alpha_k(A_k + M_k C_k)\right)\mathsf{E}[F(x_k) - F(\mathsf{x}_*)]. \qquad \square
\end{aligned}$$

Lemma B.5. [11, Lemma 3]. *If a convex function f is L-relatively smooth w.r.t. h, then for any $\eta \le \frac{1}{L}$ and $\mathsf{x}, \mathsf{y} \in \operatorname{int}\mathcal{C}$,*

$$D_f(\mathsf{x}, \mathsf{y}) \ge \frac{1}{\eta} D_{h^*}\left(\nabla h(\mathsf{x}) - \eta(\nabla f(\mathsf{x}) - \nabla f(\mathsf{y})), \nabla h(\mathsf{x})\right).$$

Lemma B.6. [11, Lemma 2]. *Let $\mathsf{x} \in \operatorname{int}\mathcal{C}$, and $g_1, g_2 \in \mathbb{R}^d$. Define the points $\mathsf{x}_1^+, \mathsf{x}_2^+, \mathsf{x}^+$ as the unique points satisfying $\nabla h(\mathsf{x}_1^+) = \nabla h(\mathsf{x}) - g_1$, $\nabla h(\mathsf{x}_2^+) = \nabla h(\mathsf{x}) - g_2$, $\nabla h(\mathsf{x}^+) = \nabla h(\mathsf{x}) - \frac{g_1+g_2}{2}$. Then*

$$\begin{aligned}
D_h(\mathsf{x}, \mathsf{x}^+) &\le \frac{1}{2}\left[D_h(\mathsf{x}, \mathsf{x}_1^+) + D_h(\mathsf{x}, \mathsf{x}_2^+)\right] \\
&= \frac{1}{2}\Big[D_{h^*}\Big(\nabla(\mathsf{x}_1^+), \nabla h(\mathsf{x})\Big) + D_{h^*}\Big(\nabla(\mathsf{x}_2^+), \nabla h(\mathsf{x})\Big)\Big].
\end{aligned}$$

For the Euclidean case, i.e. $h = \|\cdot\|^2$, we obtain the standard inequality $\|\frac{g_1+g_2}{2}\|^2 \le \frac{1}{2}\left(\|g_1\|^2 + \|g_2\|^2\right)$.

Now we recall the Bregman version of the following Euclidean variance decomposition for a r.v. x:

$$\mathsf{E}\|x\|^2 = \|\mathsf{E}[x]\|^2 + \mathsf{E}\|x - \mathsf{E}[x]\|^2.$$

Lemma B.7 (Bregman variance decomposition by [24])**.** *Let x be a random variable on $\mathbb{R}^d$. Then for any $\mathsf{u} \in \mathbb{R}^d$,*

$$\mathsf{E}[D_{h^*}(x, \mathsf{u})] = D_{h^*}(\mathsf{E}[x], \mathsf{u}) + \mathsf{E}[D_{h^*}(x, E[x])].$$

B.3 Proofs of Section 4.1

Proof of Theorem 4.6. From Lemma B.4 and since $B_k - \rho M_k \le 0$, we get

$$\begin{aligned}
(1/\alpha_k)\left(1 - \alpha_k(A_k + M_k C_k)\right)\mathsf{E}[F(x_k) - F(\mathsf{x}_*)] &\le \mathsf{E}\left[\frac{1}{\alpha_k^2}D_h(\mathsf{x}_*, x_k) + M_k\sigma_k^2\right] \\
&\quad - \mathsf{E}\left[\frac{1}{\alpha_{k+1}^2}D_h(\mathsf{x}_*, x_{k+1}) + M_{k+1}\sigma_{k+1}^2\right] \\
&\quad + \mathsf{E}[N_k].
\end{aligned}$$

That means that, by summing on $t = 0, \dots, k-1$, we obtain

$$\begin{aligned}\mathsf{E}[F(\bar{x}_k) - F(\mathsf{x}_*)] \leq \frac{(1/\alpha_0^2)\mathsf{E}[D_h(\mathsf{x}_*, x_0)] + M_0 \mathsf{E}[\sigma_0^2]}{\sum_{t=0}^{k-1}(1/\alpha_t)\left(1 - \alpha_t(A_t + M_t C_t)\right)} \\ + \sum_{t=0}^{k-1} \frac{\mathsf{E}[N_t]}{\sum_{t=0}^{k-1}(1/\alpha_t)\left(1 - \alpha_t(A_t + M_t C_t)\right)},\end{aligned}$$

where $\bar{x}_k = \displaystyle\sum_{t=0}^{k-1} \frac{(1/\alpha_t)\left(1 - \alpha_t(A_t + M_t C_t)\right)}{\sum_{t=0}^{k-1}(1/\alpha_t)\left(1 - \alpha_t(A_t + M_t C_t)\right)} x_t.$ □

Proof of Theorem 4.7. From Lemma B.4 and relative strong convexity of F, we have

$$\begin{aligned}\frac{1}{\alpha_{k+1}^2}\mathsf{E}[D_h(\mathsf{x}_*, x_{k+1})] + M_{k+1}\mathsf{E}[\sigma_{k+1}^2] &\leq \frac{1}{\alpha_k^2}\mathsf{E}[D_h(\mathsf{x}_*, x_k)] + \mathsf{E}[N_k] + (M_k + B_k - \rho M_k)\mathsf{E}[\sigma_k^2] \\ &\quad - \frac{1}{\alpha_k}\left(1 - \alpha_k(A_k + M_k C_k)\right)\mathsf{E}[D_F(x_k, \mathsf{x}_*)] \\ &\leq \frac{1}{\alpha_k^2}\mathsf{E}[D_h(\mathsf{x}_*, x_k)] + M_k\left[1 + \frac{B_k}{M_k} - \rho\right]\mathsf{E}[\sigma_k^2] + \mathsf{E}[N_k] \\ &\quad - \frac{\mu}{\alpha_k}\left(1 - \alpha_k(A_k + M_k C_k)\right)\mathsf{E}[D_h(x_k, \mathsf{x}_*)] \\ &\leq q_k\left(\frac{1}{\alpha_k^2}\mathsf{E}[D_h(\mathsf{x}_*, x_k)] + M_k\mathsf{E}[\sigma_k^2]\right) + \mathsf{E}[N_k],\end{aligned}$$

with $q_k = \max\left\{1 - \alpha_k\gamma_h\mu\left(1 - \alpha_k(A_k + M_k C_k)\right), 1 + \frac{B_k}{M_k} - \rho\right\}$. □

B.4 Proofs of Section 5

Proof of Theorem 5.2. From Proposition B.1, we have

$$\alpha_k\mathsf{E}[F(x_k) - F(\mathsf{x}_*)] \leq \mathsf{E}[D_h(\mathsf{x}_*, x_k)] - \mathsf{E}[D_h(\mathsf{x}_*, x_{k+1})] + \alpha_k^2\sigma_*^2. \tag{B.14}$$

Let $k \geq 1$. Summing from 0 up to $k-1$ and dividing both side by $\sum_{t=0}^{k-1}\alpha_t$, (B.14) gives

$$\begin{aligned}\sum_{t=0}^{k-1}\frac{\alpha_t}{\sum_{t=0}^{k-1}\alpha_t}\mathsf{E}[F(x_t) - F(\mathsf{x}_*)] &\leq \frac{1}{\sum_{t=0}^{k-1}\alpha_t}\left(D_h(\mathsf{x}_*, x_0) - \mathsf{E}[D_h(\mathsf{x}_*, x_k)]\right) + \sigma_*^2\frac{\sum_{t=0}^{k-1}\alpha_t^2}{\sum_{t=0}^{k-1}\alpha_t} \\ &\leq \frac{D_h(\mathsf{x}_*, x_0)}{\sum_{t=0}^{k-1}\alpha_t} + \sigma_*^2\frac{\sum_{t=0}^{k-1}\alpha_t^2}{\sum_{t=0}^{k-1}\alpha_t}.\end{aligned}$$

Using convexity and Jensen inequality, we finally obtain the result. □

Proof of Theorem 5.4. Proposition B.1 gives

$$\begin{aligned}(1 + \beta\alpha_k)\,\mathsf{E}[D_h(\mathsf{x}_*, x_{k+1})] &\leq \mathsf{E}[D_h(\mathsf{x}_*, x_k)] - \alpha_k\mathsf{E}[F(x_k) - F(\mathsf{x}_*)] + \alpha_k^2\sigma_*^2 \\ &\leq \mathsf{E}[D_h(\mathsf{x}_*, x_k)] + \alpha_k^2\sigma_*^2.\end{aligned}$$

So

$$\mathsf{E}[D_h(\mathsf{x}_*, x_{k+1})] \leq \frac{1}{1 + \beta\alpha_k}\mathsf{E}[D_h(\mathsf{x}_*, x_k)] + \frac{\alpha_k^2}{1 + \beta\alpha_k}\sigma_*^2.$$

Let $\alpha_k = \alpha$ and $q = \frac{1}{1+\beta\alpha}$. Summing from 0 up to $k-1$, we obtain

$$\begin{aligned}
\mathsf{E}[D_h(\mathsf{x}_*, x_k)] &\leq q^k \mathsf{E}[D_h(\mathsf{x}_*, x_0)] + q^k \sigma_*^2 \alpha^2 \sum_{t=0}^{k-1} q^{-t} \\
&= q^k \mathsf{E}[D_h(\mathsf{x}_*, x_0)] + \sigma_*^2 \alpha^2 \sum_{t=0}^{k-1} q^{k-t} \\
&\leq q^k \mathsf{E}[D_h(\mathsf{x}_*, x_0)] + \alpha^2 \frac{1}{1-q} \sigma_*^2,
\end{aligned}$$

which gives the results. □

Proof of Lemma 5.8. From Equation (4.2), we infer that $\nabla h(z_{k+1}) = \nabla h(x_k) - \alpha_k \nabla f_{i_k}(x_k) + \alpha_k e_k$. Set $\nabla h(\mathsf{x}_1^+) = \nabla h(x_k) - 2\alpha_k [\nabla f_{i_k}(x_k) - \nabla f_{i_k}(\mathsf{x}_*)]$ and $\nabla h(\mathsf{x}_2^+) = \nabla h(x_k) - 2\alpha_k(\nabla f_{i_k}(\mathsf{x}_*) - e_k)$. Then, from Lemma B.6, we know that $D_h(x_k, z_{k+1}) \leq (S_1 + S_2)/2$, where

$$\begin{aligned}
S_1 &= D_{h^*}(\nabla h(x_k) - 2\alpha_k[\nabla f_{i_k}(x_k) - \nabla f_{i_k}(\mathsf{x}_*)], \nabla h(x_k)), \\
S_2 &= D_{h^*}(\nabla h(x_k) - 2\alpha_k(\nabla f_{i_k}(\mathsf{x}_*) - e_k), \nabla h(x_k)).
\end{aligned}$$

Using the gain function in Equation (2.1), then Lemma B.5, and finally Equation (5.1), we have

$$\begin{aligned}
\mathsf{E}_k[S_1] &= \mathsf{E}_k[D_{h^*}(\nabla h(x_k) - 2\alpha_k(\nabla f_{i_k}(x_k) - \nabla f_{i_k}(\mathsf{x}_*)), \nabla h(x_k))] \\
&\leq 4L^2 \alpha_k^2 \mathsf{E}_k \left[G\left(\nabla h(x_k), \nabla h(x_k), \frac{1}{L}(\nabla f_{i_k}(x_k) - \nabla f_{i_k}(\mathsf{x}_*)) \right) \times \right. \\
&\qquad \left. D_{h^*}\left(\nabla h(x_k) - \frac{1}{L}(\nabla f_{i_k}(x_k) - \nabla f_{i_k}(\mathsf{x}_*)), \nabla h(x_k) \right) \right] \\
&\leq 4L\alpha_k^2 \mathsf{E}_k \left[G\left(\nabla h(x_k), \nabla h(x_k), \frac{1}{L}(\nabla f_{i_k}(x_k) - \nabla f_{i_k}(\mathsf{x}_*)) \right) \right] D_F(x_k, \mathsf{x}_*) \\
&\leq 4L\alpha_k^2 G_k D_F(x_k, \mathsf{x}_*).
\end{aligned}$$

We know that $-2\alpha_k(\nabla f_{i_k}(\mathsf{x}_*) - e_k) = \zeta_k - \mathsf{E}_k[\zeta_k]$. From Lemma B.7, Equation (2.1), and Equation (5.2), it follows that

$$\begin{aligned}
\mathsf{E}_k[S_2] &= \mathsf{E}_k[D_{h^*}(\nabla h(x_k) + \zeta_k - \mathsf{E}_k[\zeta_k], \nabla h(x_k))] \\
&= \mathsf{E}[D_{h^*}(\nabla h(x_k) + \zeta_k - \mathsf{E}_k[\zeta_k], \nabla h(x_k) - \mathsf{E}_k[\zeta_k]) \,|\, \mathfrak{F}_k] \\
&\quad - \mathsf{E}_k[D_{h^*}(\nabla h(x_k), \nabla h(x_k) - \mathsf{E}_k[\zeta_k])] \\
&\leq 4L^2\alpha_k^2 \mathsf{E}_k \Bigg[G\Bigg(\nabla h(x_k) - 2\alpha_k \frac{1}{n} \sum_{i=1}^n \nabla f_i(\phi_i^k), \nabla h(\phi_{i_k}^k), \frac{1}{L}\left(\nabla f_{i_k}(\phi_{i_k}^k) - \nabla f_{i_k}(\mathsf{x}_*) \right) \Bigg) \times \\
&\qquad D_{h^*}\left(\nabla h(\phi_{i_k}^k) - \frac{1}{L}\left(\nabla f_{i_k}(\phi_{i_k}^k) - \nabla f_{i_k}(\mathsf{x}_*) \right), \nabla h(\phi_{i_k}^k) \right) \Bigg] \\
&\qquad - \mathsf{E}_k[D_{h^*}(\nabla h(x_k), \nabla h(x_k) - \mathsf{E}_k[\zeta_k])] \\
&\leq 2\alpha_k^2 G_k \sigma_k^2 - \mathsf{E}_k[D_{h^*}(\nabla h(x_k), \nabla h(x_k) - \mathsf{E}_k[\zeta_k])].
\end{aligned}$$

Putting all together, we get Equation (5.3). For Equation (5.4), we have

$$
\begin{aligned}
\sigma_{k+1}^2 &= 2L^2 \mathsf{E}_{k+1}\left[D_{h^*}\left(\nabla h\left(\phi_{i_{k+1}}^{k+1}\right) - \frac{1}{L}\left(\nabla f_{i_{k+1}}\left(\phi_{i_{k+1}}^{k+1}\right) - \nabla f_{i_{k+1}}(\mathsf{x}_*)\right), \nabla h\left(\phi_{i_{k+1}}^{k+1}\right)\right)\right] \\
&= 2L^2 \frac{1}{n}\sum_{i=1}^n D_{h^*}\left(\nabla h\left(\phi_i^{k+1}\right) - \frac{1}{L}\left(\nabla f_i\left(\phi_i^{k+1}\right) - \nabla f_i(\mathsf{x}_*)\right), \nabla h\left(\phi_i^{k+1}\right)\right) \\
&= 2L^2 \frac{1}{n}\sum_{i=1}^n D_{h^*}\Big(\nabla h\left(\phi_i^k + \delta_{i,i_k}(x_k - \phi_i^k)\right) - \frac{1}{L}\left(\nabla f_i\left(\phi_i^k + \delta_{i,i_k}(x_k - \phi_i^k)\right) - \nabla f_i(\mathsf{x}_*)\right), \\
&\qquad\qquad \nabla h\left(\phi_i^k + \delta_{i,i_k}(x_k - \phi_i^k)\right)\Big).
\end{aligned}
$$

That means

$$
\begin{aligned}
&\mathsf{E}_k\left[\sigma_{k+1}^2\right] \\
&= 2L^2 \frac{1}{n}\sum_{i=1}^n \mathsf{E}_k\Big[D_{h^*}\Big(\nabla h\left(\phi_i^k + \delta_{i,i_k}(x_k - \phi_i^k)\right) - \frac{1}{L}\left(\nabla f_i\left(\phi_i^k + \delta_{i,i_k}(x_k - \phi_i^k)\right) - \nabla f_i(\mathsf{x}_*)\right), \\
&\qquad\qquad \nabla h\left(\phi_i^k + \delta_{i,i_k}(x_k - \phi_i^k)\right)\Big)\Big] \\
&= 2L^2 \frac{1}{n}\sum_{i=1}^n \frac{1}{n}\sum_{j=1}^n D_{h^*}\Big(\nabla h\left(\phi_i^k + \delta_{i,j}(x_k - \phi_i^k)\right) - \frac{1}{L}\left(\nabla f_i\left(\phi_i^k + \delta_{i,j}(x_k - \phi_i^k)\right) - \nabla f_i(\mathsf{x}_*)\right), \\
&\qquad\qquad \nabla h\left(\phi_i^k + \delta_{i,j}(x_k - \phi_i^k)\right)\Big) \\
&= 2L^2 \frac{1}{n}\sum_{i=1}^n \Big[\frac{n-1}{n} D_{h^*}\left(\nabla h\left(\phi_i^k\right) - \frac{1}{L}\left(\nabla f_i\left(\phi_i^k\right) - \nabla f_i(\mathsf{x}_*)\right), \nabla h\left(\phi_i^k\right)\right) \\
&\qquad\qquad + \frac{1}{n} D_{h^*}\left(\nabla h\left(x_k\right) - \frac{1}{L}\left(\nabla f_i\left(x_k\right) - \nabla f_i(\mathsf{x}_*)\right), \nabla h\left(x_k\right)\right)\Big] \\
&= \left(1 - \frac{1}{n}\right) 2L^2 \frac{1}{n}\sum_{i=1}^n D_{h^*}\left(\nabla h\left(\phi_i^k\right) - \frac{1}{L}\left(\nabla f_i\left(\phi_i^k\right) - \nabla f_i(\mathsf{x}_*)\right), \nabla h\left(\phi_i^k\right)\right) \\
&\qquad\qquad + 2L^2 \frac{1}{n^2}\sum_{i=1}^n D_{h^*}\left(\nabla h\left(x_k\right) - \frac{1}{L}\left(\nabla f_i\left(x_k\right) - \nabla f_i(\mathsf{x}_*)\right), \nabla h\left(x_k\right)\right) \\
&\le \left(1 - \frac{1}{n}\right)\sigma_k^2 + \frac{2L}{n} D_F(x_k, \mathsf{x}_*).
\end{aligned}
$$

The last inequality comes from Lemma B.5. □

Proof of Lemma 5.14. From Lemma B.6, we know that $D_h(x_k, z_{k+1}) \le (S_1 + S_2)/2$, where

$$
\begin{aligned}
S_1 &= D_{h^*}(\nabla h(x_k) - 2\alpha_k[\nabla f_{i_k}(x_k) - \nabla f_{i_k}(\mathsf{x}_*)], \nabla h(x_k)), \\
S_2 &= D_{h^*}(\nabla h(x_k) - 2\alpha_k(\nabla f_{i_k}(\mathsf{x}_*) - e_k), \nabla h(x_k)).
\end{aligned}
$$

Using the gain function in Equation (2.1), then Lemma B.5, and finally Equation (5.5), we have

$$\begin{aligned}
\mathsf{E}_k[S_1] &= \mathsf{E}_k[D_{h^*}(\nabla h(x_k) - 2\alpha_k(\nabla f_{i_k}(x_k) - \nabla f_{i_k}(\mathsf{x}_*)), \nabla h(x_k))] \\
&\leq 4L^2\alpha_k^2 \mathsf{E}_k \left[G\left(\nabla h(x_k), \nabla h(x_k), \frac{1}{L}(\nabla f_{i_k}(x_k) - \nabla f_{i_k}(\mathsf{x}_*))\right) \times \right. \\
&\qquad \left. D_{h^*}\left(\nabla h(x_k) - \frac{1}{L}(\nabla f_{i_k}(x_k) - \nabla f_{i_k}(\mathsf{x}_*)), \nabla h(x_k)\right)\right] \\
&\leq 4L\alpha_k^2 \mathsf{E}_k \left[G\left(\nabla h(x_k), \nabla h(x_k), \frac{1}{L}(\nabla f_{i_k}(x_k) - \nabla f_{i_k}(\mathsf{x}_*))\right)\right] D_F(x_k, \mathsf{x}_*) \\
&\leq 4L\alpha_k^2 G_k D_F(x_k, \mathsf{x}_*).
\end{aligned}$$

We know that $-2\alpha_k(\nabla f_{i_k}(\mathsf{x}_*) - e_k) = \zeta_k - \mathsf{E}_k[\zeta_k]$. From Lemma B.7, Equation (2.1), and Equation (5.6), it follows that

$$\begin{aligned}
\mathsf{E}_k[S_2] &= \mathsf{E}_k[D_{h^*}(\nabla h(x_k) + \zeta_k - \mathsf{E}_k[\zeta_k], \nabla h(x_k))] \\
&= \mathsf{E}[D_{h^*}(\nabla h(x_k) + \zeta_k - \mathsf{E}_k[\zeta_k], \nabla h(x_k) - \mathsf{E}_k[\zeta_k]) \,|\, \mathfrak{F}_k] \\
&\quad - \mathsf{E}_k[D_{h^*}(\nabla h(x_k), \nabla h(x_k) - \mathsf{E}_k[\zeta_k])] \\
&\leq 4L^2\alpha_k^2 \mathsf{E}_k \left[G\left(\nabla h(x_k) - 2\alpha_k \nabla F(u_k), \nabla h(u_k), \frac{1}{L}\left(\nabla f_{i_k}(u_k) - \nabla f_{i_k}(\mathsf{x}_*)\right)\right) \times \right. \\
&\qquad \left. D_{h^*}\left(\nabla h(u_k) - \frac{1}{L}\left(\nabla f_{i_k}(u_k) - \nabla f_{i_k}(\mathsf{x}_*)\right), \nabla h(u_k)\right)\right] \\
&\quad - \mathsf{E}_k[D_{h^*}(\nabla h(x_k), \nabla h(x_k) - \mathsf{E}_k[\zeta_k])] \\
&\leq 2\alpha_k^2 G_k \sigma_k^2 - \mathsf{E}_k[D_{h^*}(\nabla h(x_k), \nabla h(x_k) - \mathsf{E}_k[\zeta_k])].
\end{aligned}$$

Putting all together, we get Equation (5.7). For Equation (5.4), we have

$$\begin{aligned}
\sigma_{k+1}^2 &= 2L^2 \mathsf{E}_{k+1}\left[D_{h^*}\left(\nabla h(u_{k+1}) - \frac{1}{L}(\nabla f_{i_{k+1}}(u_{k+1}) - \nabla f_{i_{k+1}}(\mathsf{x}_*)), \nabla h(u_{k+1})\right)\right], \\
&= 2L^2 \mathsf{E}_{k+1}\left[D_{h^*}\left(\nabla h\left((1-\varepsilon_k)u_k + \varepsilon_k x_k\right) - \frac{1}{L}\left(\nabla f_{i_{k+1}}\left((1-\varepsilon_k)u_k + \varepsilon_k x_k\right) - \nabla f_{i_{k+1}}(\mathsf{x}_*)\right),\right.\right. \\
&\qquad \left.\left. \nabla h\left((1-\varepsilon_k)u_k + \varepsilon_k x_k\right)\right)\right] \\
&= 2L^2 \frac{1}{n}\sum_{i=1}^n D_{h^*}\left(\nabla h\left((1-\varepsilon_k)u_k + \varepsilon_k x_k\right) - \frac{1}{L}\left(\nabla f_i\left((1-\varepsilon_k)u_k + \varepsilon_k x_k\right) - \nabla f_i(\mathsf{x}_*)\right),\right. \\
&\qquad \left. \nabla h\left((1-\varepsilon_k)u_k + \varepsilon_k x_k\right)\right).
\end{aligned}$$

That means

$$\begin{aligned}
\mathsf{E}_k\left[\sigma_{k+1}^2\right] &= 2L^2 \frac{1}{n}\sum_{i=1}^n \mathsf{E}_k\left[D_{h^*}\left(\nabla h\left((1-\varepsilon_k)u_k + \varepsilon_k x_k\right) - \frac{1}{L}\left(\nabla f_i\left((1-\varepsilon_k)u_k + \varepsilon_k x_k\right) - \nabla f_i(\mathsf{x}_*)\right),\right.\right. \\
&\qquad \left.\left. \nabla h\left((1-\varepsilon_k)u_k + \varepsilon_k x_k\right)\right)\right] \\
&= (1-p)\sigma_k^2 + 2pL^2 \frac{1}{n}\sum_{i=1}^n D_{h^*}\left(\nabla h(x_k) - \frac{1}{L}\left(\nabla f_i(x_k) - \nabla f_i(\mathsf{x}_*)\right), \nabla h(x_k)\right) \\
&\leq (1-p)\sigma_k^2 + 2pLD_F(x_k, \mathsf{x}_*).
\end{aligned}$$

The last inequality comes from Lemma B.5. □

Proof of Lemma A.2. Thanks to Lemma B.6, we know that $D_h(x_k, z_{k+1}) \leq (S_1 + S_2)/2$, where

$$S_1 = D_{h^*}(\nabla h(x_k) - 2\alpha_k[\nabla f_{i_k}(x_k) - \nabla f_{i_k}(\mathsf{x}_*)], \nabla h(x_k)),$$
$$S_2 = D_{h^*}(\nabla h(x_k) - 2\alpha_k(\nabla f_{i_k}(\mathsf{x}_*) - e_k), \nabla h(x_k)).$$

Using the gain function in Equation (2.1), then Lemma B.5, and finally Equation (A.1), we have

$$\begin{aligned}
\mathsf{E}[S_1 \mid \mathfrak{F}_{s,k}] &= \mathsf{E}[D_{h^*}(\nabla h(x_k) - 2\alpha_k(\nabla f_{i_k}(x_k) - \nabla f_{i_k}(\mathsf{x}_*)), \nabla h(x_k)) \mid \mathfrak{F}_{s,k}] \\
&\leq 4L^2\alpha_k^2 \mathsf{E}\left[G\left(\nabla h(x_k), \nabla h(x_k), \frac{1}{L}(\nabla f_{i_k}(x_k) - \nabla f_{i_k}(\mathsf{x}_*))\right) \times \right. \\
&\qquad \left. D_{h^*}\left(\nabla h(x_k) - \frac{1}{L}(\nabla f_{i_k}(x_k) - \nabla f_{i_k}(\mathsf{x}_*)), \nabla h(x_k)\right) \mid \mathfrak{F}_{s,k}\right] \\
&\leq 4L\alpha_k^2 \mathsf{E}\left[G\left(\nabla h(x_k), \nabla h(x_k), \frac{1}{L}(\nabla f_{i_k}(x_k) - \nabla f_{i_k}(\mathsf{x}_*))\right) \mid \mathfrak{F}_{s,k}\right] D_F(x_k, \mathsf{x}_*) \\
&\leq 4L\alpha_k^2 G_k D_F(x_k, \mathsf{x}_*).
\end{aligned}$$

We know that $-2\alpha_k(\nabla f_{i_k}(\mathsf{x}_*) - e_k) = \zeta_k - \mathsf{E}_k[\zeta_k]$. From Lemma B.7, Equation (2.1), and Equation (A.2), it follows that

$$\begin{aligned}
\mathsf{E}[S_2 \mid \mathfrak{F}_{s,k}] &= \mathsf{E}[D_{h^*}(\nabla h(x_k) + \zeta_k - \mathsf{E}_k[\zeta_k], \nabla h(x_k)) \mid \mathfrak{F}_{s,k}] \\
&= \mathsf{E}[D_{h^*}(\nabla h(x_k) + \zeta_k - \mathsf{E}_k[\zeta_k], \nabla h(x_k) - \mathsf{E}_k[\zeta_k])] \\
&\quad - \mathsf{E}_k[D_{h^*}(\nabla h(x_k), \nabla h(x_k) - \mathsf{E}_k[\zeta_k]) \mid \mathfrak{F}_{s,k}] \\
&\leq 4L^2\alpha_k^2 \mathsf{E}\left[G\left(\nabla h(x_k) - 2\alpha_k \nabla F(\tilde{x}_s), \nabla h(\tilde{x}_s), \frac{1}{L}(\nabla f_{i_k}(\tilde{x}_s) - \nabla f_{i_k}(\mathsf{x}_*))\right) \times \right. \\
&\qquad \left. D_{h^*}\left(\nabla h(\tilde{x}_s) - \frac{1}{L}(\nabla f_{i_k}(\tilde{x}_s) - \nabla f_{i_k}(\mathsf{x}_*)), \nabla h(\tilde{x}_s)\right) \mid \mathfrak{F}_{s,k}\right] \\
&\quad - \mathsf{E}[D_{h^*}(\nabla h(x_k), \nabla h(x_k) - \mathsf{E}_k[\zeta_k \mid \mathfrak{F}_{s,k}]) \mid \mathfrak{F}_{s,k}] \\
&\leq 2\alpha_k^2 G_k \sigma_k^2 - \mathsf{E}[D_{h^*}(\nabla h(x_k), \nabla h(x_k) - \mathsf{E}[\zeta_k \mid \mathfrak{F}_{s,k}]) \mid \mathfrak{F}_{s,k}].
\end{aligned}$$

Putting all together give the results. □

Proof of Theorem A.5. Proposition B.1 and Lemma B.7 give

$$\begin{aligned}
\mathsf{E}[D_h(\mathsf{x}_*, x_{k+1})] &\leq D_h(\mathsf{x}_*, x_k) + \alpha_k^2 B_k \sigma_k^2 \\
&\qquad - \alpha_k\,[1 - \alpha_k A_k]\,\mathsf{E}[F(x_k) - F(\mathsf{x}_*)] \\
&= D_h(\mathsf{x}_*, x_k) - \alpha_k\,[1 - \alpha_k A_k]\,\mathsf{E}[F(x_k) - F(\mathsf{x}_*)] \\
&\qquad + 2L^2\alpha_k^2 B_k \mathsf{E}\left[D_{h^*}\left(\nabla h(\tilde{x}_s) - \frac{1}{L}(\nabla f_{i_k}(\tilde{x}_s) - \nabla f_{i_k}(\mathsf{x}_*)), \nabla h(\tilde{x}_s)\right) \mid \mathfrak{F}_{s,k}\right] \\
&\leq D_h(\mathsf{x}_*, x_k) - \alpha_k\,[1 - \alpha_k A_k]\,\mathsf{E}[F(x_k) - F(\mathsf{x}_*)] \\
&\qquad + 2L\alpha_k^2 B_k \mathsf{E}[D_F(\tilde{x}_s, \mathsf{x}_*)].
\end{aligned}$$

In the last inequality, we used Lemma B.5. By taking total expectation, it follows

$$\begin{aligned}
\mathsf{E}[D_h(\mathsf{x}_*, x_{k+1})] &\leq \mathsf{E}[D_h(\mathsf{x}_*, x_k)] + 2\alpha_k^2 B_k L \mathsf{E}[D_F(\tilde{x}_s, \mathsf{x}_*)] \\
&\qquad - \alpha_k\,[1 - \alpha_k A_k]\,\mathsf{E}[F(x_k) - F(\mathsf{x}_*)].
\end{aligned}$$

Then by summing over $k = 0, \ldots, m-1$, taking the total expectation and recalling Remark A.4, we obtain

$$\begin{aligned}
\mathsf{E}[D_h(\mathsf{x}_*, x^m)] + \alpha\,[1-\alpha A]\,m\mathsf{E}[D_F(\tilde{x}_{s+1}, \mathsf{x}_*)] &\le \mathsf{E}[D_h(\mathsf{x}_*, x_0)] + 2\alpha^2 BLm\mathsf{E}[D_F(\tilde{x}_s, \mathsf{x}_*)] \\
&= \mathsf{E}[D_h(\mathsf{x}_*, \tilde{x}_s)] + 2\alpha^2 BLm\mathsf{E}[D_F(\tilde{x}_s, \mathsf{x}_*)] \\
&\le \frac{1}{\mu}\mathsf{E}[D_F(\mathsf{x}_*, \tilde{x}_s)] + 2\alpha^2 BLm\mathsf{E}[D_F(\tilde{x}_s, \mathsf{x}_*)] \\
&\le \frac{1}{\gamma_h \mu}\mathsf{E}[D_F(\tilde{x}_s, \mathsf{x}_*)] + 2\alpha^2 BLm\mathsf{E}[D_F(\tilde{x}_s, \mathsf{x}_*)] \\
&= \left(\frac{1}{\gamma_h \mu} + 2\alpha^2 BLm\right)\mathsf{E}[D_F(\tilde{x}_s, \mathsf{x}_*)].
\end{aligned}$$

In the first inequality, we used the fact that

$$\sum_{k=0}^{m-1} F(x_k) = m \sum_{k=0}^{m-1} \frac{1}{m} F(x_k) = m \sum_{\xi=0}^{m-1} \frac{1}{m} F\left(\textstyle\sum_{k=0}^{m-1} \delta_{k,\xi} x_k\right) = m\mathsf{E}[F(\tilde{x}_{s+1}) \,|\, \mathfrak{F}_{s,m-1}].$$

It still holds by choosing $\tilde{x}_{s+1} = \sum_{k=0}^{m-1} \frac{1}{m} x_k$, in Algorithm A.1, and using Jensen inequality to lower bound $\sum_{k=0}^{m-1} F(x_k)$ by $mF(\tilde{x}_{s+1})$. We finally have

$$\begin{aligned}
\mathsf{E}[D_F(\tilde{x}_{s+1}, \mathsf{x}_*)] &\le (\alpha\,(1-\alpha A)\,m)^{-1}\left(\frac{1}{\gamma_h \mu} + 2\alpha^2 BLm\right)\mathsf{E}[D_F(\tilde{x}_s, \mathsf{x}_*)] \\
&= \left(\frac{1}{\gamma_h \mu \alpha\,(1-\alpha A)\,m} + \frac{2\alpha BL}{1-\alpha A}\right)\mathsf{E}[D_F(\tilde{x}_s, \mathsf{x}_*)].
\end{aligned}$$

Replacing the constants by their values gives the result. □

C Full results of the numerical experiments

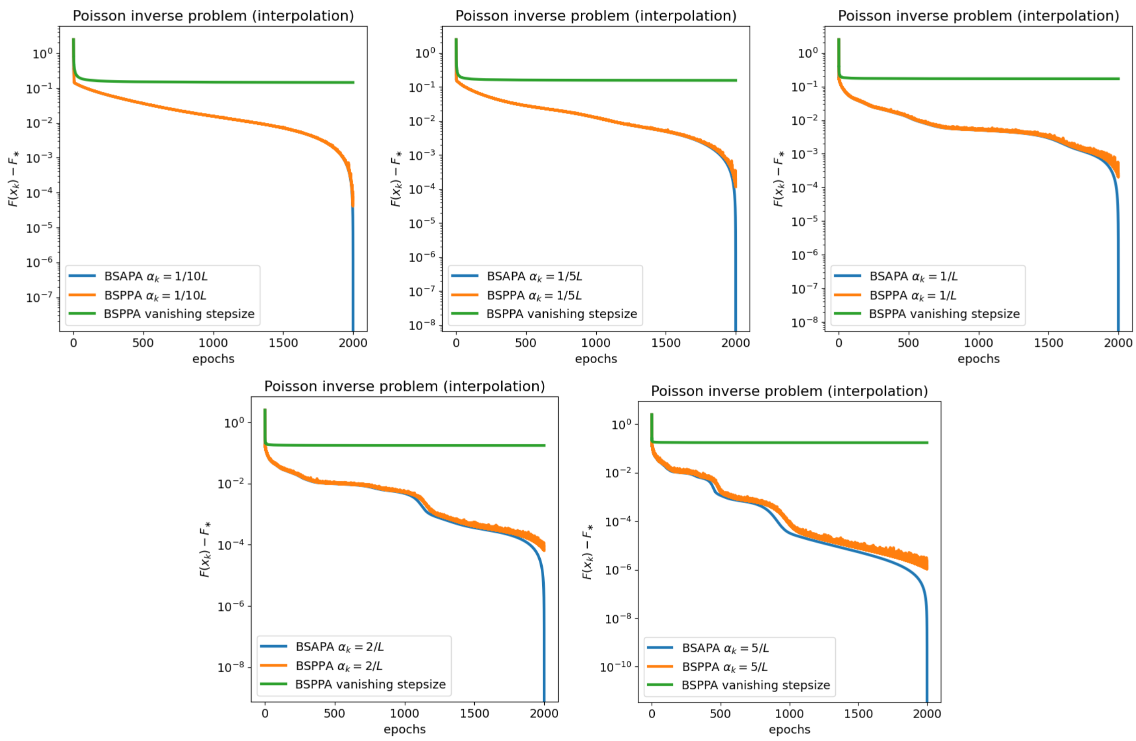

Figure 3: Poisson linear inverse problem *(interpolation case)* with different stepsizes.

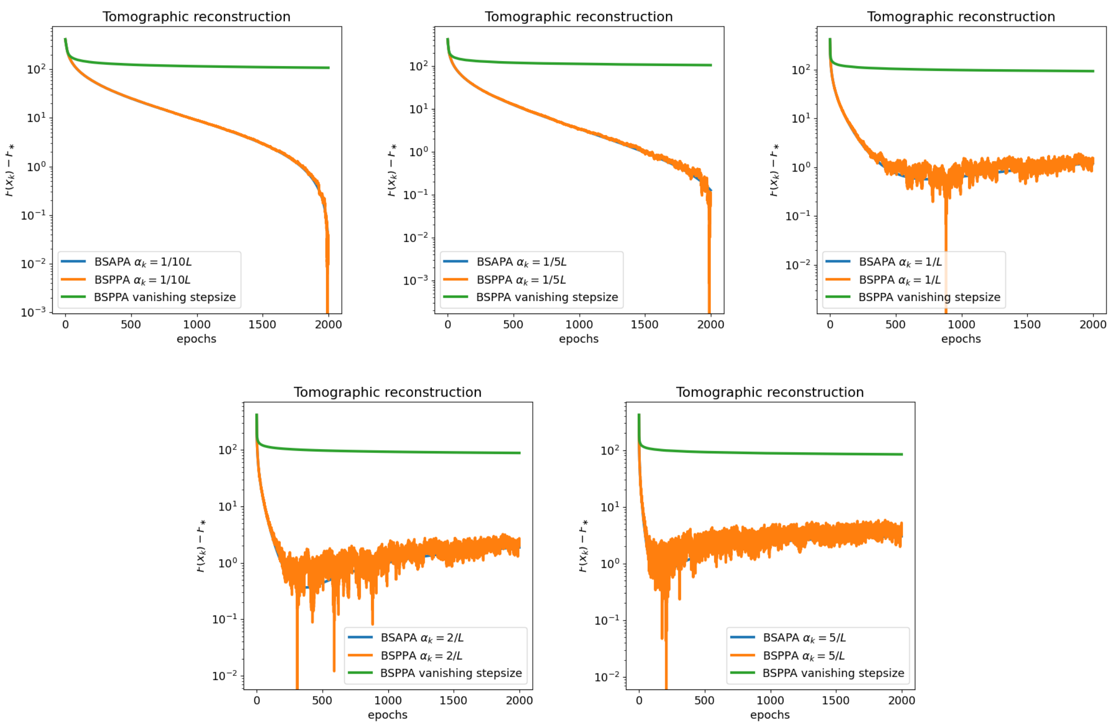

Figure 4: Tomographic reconstruction with different stepsizes.

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