Inertial Quasi-Newton Methods for Monotone Inclusion: Efficient Resolvent Calculus and Primal-Dual Methods

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Abstract

We introduce an inertial quasi-Newton Forward-Backward Splitting Algorithm to solve a class of monotone inclusion problems. While the inertial step is computationally cheap, in general, the bottleneck is the evaluation of the resolvent operator. A change of the metric makes its computation hard even for (otherwise in the standard metric) simple operators. In order to fully exploit the advantage of adapting the metric, we develop a new efficient resolvent calculus for a low-rank perturbed standard metric, which accounts exactly for quasi-Newton metrics. Moreover, we prove the convergence of our algorithms, including linear convergence rates in case one of the two considered operators is strongly monotone. Beyond the general monotone inclusion setup, we instantiate a novel inertial quasi-Newton Primal-Dual Hybrid Gradient Method for solving saddle point problems. The favourable performance of our inertial quasi-Newton PDHG method is demonstrated on several numerical experiments in image processing.

1 Introduction

Nowadays, convex optimization appears in many modern disciplines, especially when dealing with datasets of large scale. There is a strong need of efficient optimization schemes. Unfortunately, the high dimension of the problems at hand makes the use of second order methods intractable. A promising alternative are quasi-Newton type methods, which aim for exploiting cheap and accurate first order approximations of the second order information. In particular, the so-called limited memory quasi-Newton method has proved very successful for solving unconstrained large scale problems. However, many practical problems in machine learning, image processing or statistics more naturally have constraints or are non-smooth by construction.

A problem structure that can cover a broad class of non-smooth problems in these applications is the monotone inclusion problem on a real Hilbert space $\mathcal{H}$ as the following:

$$\text{find } x \in \mathcal{H} \ \text{ such that } \ (A+B)x \ni 0\,, \tag{1}$$

where $A\colon \mathcal{H} \rightrightarrows \mathcal{H}$ is a maximally monotone operator and $B\colon \mathcal{H} \to \mathcal{H}$ is a single-valued β cocoercive operator. As a special case (1) comprises the setting of minimization problems of the form $\min_{x\in\mathcal{H}} f(x) + g(x)$ with a proper lower semi-continuous convex function f and convex function g with Lipschitz continuous gradient by setting $A = \partial f$ and $B = \nabla g$.

A fundamental algorithmic scheme to tackle the problem class (1) is Forward-Backward Splitting (FBS). However, this algorithm can be quite slow for ill-conditioned problems, for which one would like to exploit the second order information to adapt to the local geometry of the problem. As a computationally affordable approximation in this paper, we propose a quasi-Newton variant that takes advantage from a variable metric that is computed using first order information only. We manage to remedy the main computational bottlenecks for these type of approach by developing an efficient low-rank variable metric resolvent calculus.

Our approach is inspired by the proximal quasi-Newton method in [7]. We extend the framework proposed in [7] to the resolvent setting with a $M \pm$ rank-r symmetric positive definite variable metric. We first study the convergence of two variants of FBS algorithm with respect to this type of metrics. One variant uses inertial step to accelerate its convergence and the other uses relaxation to guarantee convergence with mild assumption. In analogy to the proximal calculus proposed in [7], we develop a resolvent calculus that allows for a efficient evaluation of resolvent operators with respect to this type of metrics by splitting the evaluation into two computationally simple steps: calculating a resolvent operator with respect to a simple metric M and solving a low dimensional root finding problem. This allows for using popular quasi-Newton methods, such as the limited memory SR1 or BFGS method.

In order to exploit the power and to illustrate the variety of problems that can be solved via the framework as (1), the developed algorithms are instantiated for the following saddle point problem, which include quasi-Newton Primal-Dual methods and enjoy an enormous number of potential applications:

$$\min_{x \in \mathcal{H}_1} \max_{y \in \mathcal{H}_2} \langle Kx, y \rangle + g(x) + G(x) - f(y) - F(y) \,, \tag{2}$$

where g and f are lower semi-continuous convex functions, G and F are convex, differentiable with Lipschitz gradients and K is a bounded linear operator. The numerical performance of our algorithms is tested on several numerical experiments and demonstrates a clear improvement when our quasi-Newton methods are used.

1.1 Related Works

Smooth quasi-Newton. Quasi-Newton methods are widely studied and used for optimization with objective functions sufficiently smooth [35]. Their motivation is to build a cheap approximation to the Newton's Method. If the approximation of the second order information is given by a positive definite matrix, quasi-Newton methods can be interpreted as adapting the metric of the space locally to the objective function. The success of these methods requires a good approximation of the second order information (Hessian) of the objective by using the first order information, which is still actively researched. Recently, [30] proposed to select greedily some basis vectors instead of using the difference of successive iterates for updating the Hessian approximations. Based on their works, [24] constructed an approximation of the indefinite Hessian of a twice differentiable function. All methods mentioned above require sufficient smoothness of the objective functions.

Non-smooth quasi-Newton. A broad class of optimization problems is interpreted as a composition of a smooth function f and a non-smooth function h. To deal with the non-smoothness of h efficiently, many authors consider a combination of FBS with the quasi-Newton methods. By using the forward-backward envelope, [27, 33] reinterpreted the FBS algorithm as a variable metric gradient method for a smooth optimization problem in order to apply the classic Newton or quasi-Newton method. For non-smooth function g as simple as an indicator function of a convex set, [32, 31] proposed an elegant method named projected quasi-Newton algorithm (PQN) which, however, requires either solving a subproblem or a diagonal metric. [23] extended PQN to a more general setting as long as the proximal operator with respect to h is simple to compute. For a class of low-rank perturbed metrics, [6, 7] developed a proximal quasi-Newton method with a root finding problem as the subproblem which can be solved easily and efficiently. This method can be extended to the nonconvex setting [19]. Based on [6, 7], [20] incorporated a limited-memory quasi-Newton update. Meanwhile, [21] developed a different algorithm to evaluate the proximal operator of the separable l_1 norm with respect to a low-rank metric $V = M - UU^*$. We extend the quasi-Newton approach of [6, 7] from the nonsmooth convex minimization setting to monotone inclusion problems of type (1). More related works for the minimization setting are intensively discussed in [7]. Therefore, we focus here on approaches to solve the monotone inclusion problem. It is generic to use a variable metric for solving a monotone inclusion problem, which is discussed in [12, 13]. Its convergence relies on quasi-Fejér monotonicity [12]. However the efficient calculation of the resolvent is left as an open problem. Our approaches use a variable metric to obtain a quasi-Newton method with an efficient resolvent calculus. From a different perspective, [2] considered incorporating Newton methods by introducing dynamic system with Hessian damping. Later, in [1, 3], time discretization of some dynamics gives new insight into the Newton's method for solving monotone inclusions which is different from variable metric approaches. Recently, there is a new approach that [22] proposed a generalized damped Newton type which is based on second-order subdifferentials.

PDHG. Primal-Dual Hybrid Gradient (PDHG) is widely used for solving saddle point problems of the form (2). PDHG can be interpreted as a proximal point algorithm [17] applied on a monotone inclusion problem with a fixed metric. Based on this idea, [26] proposed an inertial FBS method applied to the sum of set-valued operators from which a generalization of PDHG method is derived. Also based on similar ideas, [28] considers diagonal preconditioning to accelerate PDHG. Their method can be regarded as using a fixed blocked matrix as a metric. Later, [25] considers non-diagonal preconditioning and pointed out if a special preconditioner is chosen, that kind of preconditioned PDHG method will be a special form of the linearized ADMM. Their method requires an inner loop due to the non-diagonal preconditioning. [16] introduces an adaptive PDHG scheme which can also be understood as using a variable metric with step size tuned automatically. However, [16] considered changes on the diagonal of the metric. Our resolvent calculus in Section 4 provides another possibility to change metrics at elements off the diagonal to deal with resolvent operators.

2 Preliminaries

Let us recall some essential notations and definitions. Let $\mathcal{H}$ be a Hilbert space equipped with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$. An operator $K \in \mathcal{B}(\mathcal{D},\mathcal{H})$ is a linear bounded mapping from a Hilbert space $\mathcal{D}$ to $\mathcal{H}$. We abbreviate $\mathcal{B}(\mathcal{H},\mathcal{H})$ to $\mathcal{B}(\mathcal{H})$. The adjoint of $M \in \mathcal{B}(\mathcal{H})$ is denoted by M^*. We define $\mathcal{S}(\mathcal{H}) := \{M \in \mathcal{B}(\mathcal{H}) | M = M^*\}$ and the identity operator by $I \in \mathcal{S}(\mathcal{H})$. Without ambiguity, we also use the notation $\|M\|$ for the operator norm of $M \in \mathcal{S}(\mathcal{H})$ with respect to $\|\cdot\|$. The partial ordering on $\mathcal{S}(\mathcal{H})$ is given by

$$(\forall U \in \mathcal{S}(\mathcal{H}))(\forall V \in \mathcal{S}(\mathcal{H})) \colon \quad U \succeq V \iff (\forall x \in \mathcal{H})\, \langle Ux, x\rangle \geq \langle Vx, x\rangle\,. \tag{3}$$

For $\sigma \in [0,+\infty)$, we introduce $\mathcal{S}_\sigma(\mathcal{H}) := \{U \in \mathcal{S}(\mathcal{H}) | U \succeq \sigma I)\}$. Similarly, we introduce $\mathcal{S}_{++}(\mathcal{H}) := \{U \in \mathcal{S}(\mathcal{H}) | U \succ 0\}$. The norm $\|\cdot\|_M$ is defined by $\sqrt{\langle M\cdot,\cdot\rangle}$ for some $M \in \mathcal{S}_{++}(\mathcal{H})$.

A set valued operator $A\colon \mathcal{H} \rightrightarrows \mathcal{H}$ is defined by its graph

$$\operatorname{Graph} A := \{(x,y) \in \mathcal{H} | x \in \operatorname{Dom}(A), y \in Ax\}\,,$$

and has a domain given by

$$\operatorname{Dom}(A) := \{x \in \mathcal{H} | Ax \neq \emptyset\}.$$

Given two set-valued operators $A, B\colon \mathcal{H} \rightrightarrows \mathcal{H}$, we define $A + B\colon \mathcal{H} \rightrightarrows \mathcal{H}$ as follows:

$$\begin{aligned} &\operatorname{Dom}(A+B) = \operatorname{Dom}(A) \cap \operatorname{Dom}(B)\,,\\ &\quad (A+B)x = Ax + Bx := \{y \in \mathcal{H} | \exists y_1 \in \mathcal{H}, \exists y_2 \in \mathcal{H} \text{ such that } y = y_1 + y_2\}\,. \end{aligned}$$

The inverse of A is denoted by A^{-1} given by $A^{-1}(y) := \{x \in \mathcal{H} | y \in Ax\}$ and the zero set of A is denoted by $zer(A+B) := \{x \in \mathcal{H} | (A+B)x \ni 0\}$. We call A is γ_A-strongly monotone with respect to norm $\|\cdot\|$ and $\gamma_A \geq 0$ if $\langle x-y, u-v\rangle \geq \gamma_A \|x-y\|^2$ for any pair $(x,u)\,,(y,v) \in \operatorname{Graph} A$ and $\gamma_A = 0$ if just monotone. The resolvent of $A\colon \mathcal{H} \rightrightarrows \mathcal{H}$ with respect to metric $M \in \mathcal{S}_{++}(\mathcal{H})$ is defined as

$$J_A^M := (I + M^{-1}A)^{-1} \quad \text{and we set } J_A := J_A^I \text{ for the identity metric } I\,, \tag{4}$$

which, as shown for example in [5], enjoys the following properties.

Proposition 2.1. *Let $A\colon \mathcal{H} \rightrightarrows \mathcal{H}$ be maximally monotone, $M \in \mathcal{S}_{++}(\mathcal{H})$ and $y \in \mathcal{H}$. Then, the following holds*

$$y = J_{\gamma A}^M(x) \iff x \in y + \gamma M^{-1}Ay \iff x - y \in \gamma M^{-1}Ay \iff (y, \gamma^{-1}M(x-y)) \in \operatorname{Graph} A\,. \tag{5}$$

Proposition 2.2. *Let $A : \mathcal{H} \rightrightarrows \mathcal{H}$ be maximally monotone. Then for every sequence $(x_k, u_k)_{k\in\mathbb{N}}$ in $\operatorname{Graph} A$ and every $(x,u) \in \mathcal{H}\times\mathcal{H}$, if $x_k \rightharpoonup x$ and $u_k \to u$, then $(x,u) \in \operatorname{Graph} A$.*

Lemma 2.3. *Let $T\colon \mathcal{H} \rightrightarrows \mathcal{H}$ be a maximally monotone operator and let $M \in \mathcal{S}_{++}(\mathcal{H})$. Then, for any $z \in \mathcal{H}$, we have $J_T^M(z) = M^{-1/2} \circ J_{M^{-1/2}TM^{-1/2}} \circ M^{1/2}(z)$*

Proof. See Appendix A.1. ∎

Lemma 2.4. *If A is γ_A-strongly monotone and $\gamma_A \geq 0$, then J_A^M is Lipschitz continuous with respect to $\|\cdot\|_M$ with constant $1/(1+\frac{\gamma_A}{C}) \in (0,1]$ for any C satisfying $\|M\| \leq C < \infty$.*

Proof. See Appendix A.2. □

Proposition 2.5 (Variable Metric quasi-Fejér monotone sequence [12]). *Let $\sigma \in (0, +\infty)$, let $\varphi\colon [0, +\infty) \to [0, +\infty)$ be strictly increasing and such that $\lim_{t\to+\infty} \varphi(t) = +\infty$, let $(M_k)_{k\in\mathbb{N}}$ be in $\mathcal{S}_\sigma(\mathcal{H})$, let C be a nonempty subset of $\mathcal{H}$, and let $(x_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{H}$ such that*

$$(\exists(\eta_k)_{k\in\mathbb{N}} \in \ell^1_+(\mathbb{N}))(\forall z \in C)(\exists(\epsilon_k)_{k\in\mathbb{N}} \in \ell^1_+(\mathbb{N}))(\forall k \in \mathbb{N})\colon \\ \varphi(\|x_{k+1} - z\|_{M_{k+1}}) \le (1 + \eta_k)\varphi(\|x_k - z\|_{M_k}) + \epsilon_k\,. \tag{6}$$

(a) Then $(x_k)_{k\in\mathbb{N}}$ is bounded and, for every $z \in C$, $(\|z_k - z\|_{M_k})_{k\in\mathbb{N}}$ converges.

(b) If additionally, there exists $M \in \mathcal{S}_\sigma(\mathcal{H})$ such that $M_k \to M$ pointwise, as is the case when

$$\sup_{k\in\mathbb{N}} \|M_k\| < +\infty \quad \text{and} \quad (\exists(\eta_k)_{k\in\mathbb{N}} \in \ell^1_+(\mathbb{N}))(\forall k \in N)\colon\ (1 + \eta_k)M_k \succeq M_{k+1}\,, \tag{7}$$

then $(x_k)_{k\in\mathbb{N}}$ converges weakly to a point in C if and only if every weak sequential cluster point of $(x_k)_{k\in\mathbb{N}}$ lies in C.

A key result for our resolvent calculus in Section 4 is the following abstract duality principle.

Lemma 2.6 (A duality result for operators [4]). *Let $A\colon \mathcal{H} \rightrightarrows \mathcal{H}$ be an operator such that A^{-1} is single-valued and let $B\colon \mathcal{H} \to \mathcal{H}$ be a single-valued operator. Then, the following holds for $x, u \in \mathcal{H}$:*

$$\begin{cases} 0 \in Ax + Bx \\ 0 \in B^{-1}u - A^{-1}(-u) \end{cases} \iff \begin{cases} x \in B^{-1}u \\ -u \in Ax \end{cases} \iff \begin{cases} Bx = u \\ x = A^{-1}(-u) \end{cases}\,. \tag{8}$$

Moreover, if there exists $x \in \mathcal{H}$ such that $0 \in Ax + Bx$ or there exists $u \in \mathcal{H}$ such that $0 \in B^{-1}u - A^{-1}(-u)$, then there exists a unique primal-dual pair (x, u) that satisfies the equivalent conditions above.

We also need the following lemma which was stated as [29, Lemma 2.2.2].

Lemma 2.7. *Let $C_k \ge 0$ and let*

$$\begin{aligned} &C_{k+1} \le (1 + \nu_k)C_k + \zeta_k, \quad \nu_k \ge 0,\ \zeta_k \ge 0, \\ &\sum_{k\in\mathbb{N}} \nu_k < \infty, \quad \sum_{k\in\mathbb{N}} \zeta_k < \infty. \end{aligned} \tag{9}$$

Then, $C_k \to C \ge 0$ and $C < +\infty$.

3 Inertial Quasi-Newton Forward-Backward Splitting for Monotone Inclusion

3.1 Problem setting

Let $A\colon \mathcal{H} \rightrightarrows \mathcal{H}$ be a maximally and γ_A-strongly monotone operator with $\gamma_A \ge 0$ with respect to norm $\|\cdot\|$ and let $B\colon \mathcal{H} \to \mathcal{H}$ be a single-valued, β cocoercive and γ_B-strongly monotone operator with $\gamma_B \ge 0$ with respect to norm $\|\cdot\|$. We consider the following monotone inclusion problem:

$$\text{find } x \in \mathcal{H} \ \text{ such that } \ Ax + Bx \ni 0\,. \tag{10}$$

Note that by setting $\gamma_A = 0$ or $\gamma_B = 0$, we include the general case of monotone operators that are not necessarily strongly monotone. In this paper, we propose two variants of an efficiently implementable quasi-Newton Forward-Backward Splitting (FBS) algorithm in Algorithm 1 and Algorithm 2 to solve (10). The main update step is a variable metric FBS step in both algorithms as shown in (12) and (13), i.e., a forward step with respect to the cocoercive operator B, followed by a proximal point step (computation of the resolvent) with respect to the maximally monotone operator A, both evaluated in an iteration dependent metric M_k that is inspired by quasi-Newton methods and therefore adapts to the local geometry of the problem. In contrast to the related works, as discussed in Section 1.1, we emphasize the importance of efficiently implementable resolvent operators (see Section 4). The two variants allow for more or less flexibility for the choice of the metric (see Section 5). Both variants account for potential (numerical) errors in the evaluation of the forward-backward step. Algorithm 1 combines a FBS step with an additional inertial step which has the potential of accelerating the convergence, as we illuminate in our numerical experiments in Section 7. It is a generalization of the algorithms in [10, 26] to a quasi-Newton variant. In [26], Lorenz and Pock proposed an inertial Forward-Backward Splitting algorithm but with a fixed metric which is different from our Algorithm 1 which uses a variable metric. Algorithm 2 combines FBS with a relaxation step in (iii) which yields convergence under mild assumptions on the metric. It generalizes the correction step introduced in [17] which can be retrieved by metric $M_k \equiv M$ and $B \equiv 0$.

Algorithm 1 Inertial quasi-Newton Forward-Backward Splitting

Require: $N \geq 0$, $(\|\epsilon_k\|)_{k\in\mathbb{N}} \in \ell^1_+(\mathbb{N})$
Initialization: $x_0 \in \mathcal{H}$
Update for $k = 0, \cdots, N$:

(i) Compute M_k according to a quasi-Newton framework.

(ii) Compute the inertial step (extrapolation step):

$$\bar{z}_k \leftarrow z_k + \alpha_k(z_k - z_{k-1}), \tag{11}$$

(iii) and the forward-backward step:

$$z_{k+1} \leftarrow J_A^{M_k}(\bar{z}_k - M_k^{-1}B\bar{z}_k) + \epsilon_k. \tag{12}$$

End

Algorithm 2 Quasi-Newton Forward-Backward Splitting with Relaxation

Require: $N \geq 0$, $(\|\epsilon_k\|)_{k\in\mathbb{N}} \in \ell^1_+(\mathbb{N})$
Initialization: $x_0 \in \mathcal{H}$
Update for $k = 0, \cdots, N$

(i) Compute M_k according to a quasi-Newton framework.

(ii) Compute the forward-backward step:

$$\tilde{z}_k \leftarrow J_A^{M_k}(z_k - M_k^{-1}Bz_k) + \epsilon_k, \tag{13}$$

(iii) and relaxation step:

$$t_k = \frac{\langle z_k - \tilde{z}_k, (M_k - B)(z_k - \tilde{z}_k)\rangle}{2\|(M_k - B)(z_k - \tilde{z}_k)\|^2}, \tag{14}$$

$$z_{k+1} \leftarrow z_k - t_k[(M_k - B)(z_k - \tilde{z}_k)]. \tag{15}$$

End

3.2 Convergence

In this subsection, we prove the convergence of Algorithm 1 and Algorithm 2. The implementation details for the specific quasi-Newton features are deferred to the upcoming section.

3.2.1 Algorithm 1: Inertial Quasi-Newton Forward-Backward Splitting

The following convergence result is a generalization of [11, Theorem 3.1] to an inertial version of variable metric Forward-Backward Splitting.

Assumption 1. *Let $\sigma \in (0, +\infty)$. $(M_k)_{k\in\mathbb{N}}$ is a sequence in $\mathcal{S}_\sigma(\mathcal{H})$ such that*

$$\begin{cases} C := \sup_{k\in\mathbb{N}} ||M_k|| < \infty\,, \\ (\exists(\eta_k)_{k\in\mathbb{N}} \in \ell^1_+(\mathbb{N}))(\forall k \in \mathbb{N}): \quad (1+\eta_k)M_k \succeq M_{k+1}\,, \end{cases} \tag{16}$$

and $M_k - \frac{1}{2\beta}I \in \mathcal{S}_\epsilon(\mathcal{H})$ for all $k \in \mathbb{N}$ and some $\epsilon > 0$. Moreover, $(\epsilon_k)_{k\in\mathbb{N}}$ is a sequence in $\mathcal{H}$ such that $\sum_{k\in\mathbb{N}} \|\epsilon_k\| < +\infty$.

Theorem 3.1. *Consider Problem (10) and let the sequence $(z_k)_{k\in\mathbb{N}}$ be generated by Algorithm 1 where Assumption 1 holds and $(\alpha_k)_{k\in\mathbb{N}}$ are selected such that $\alpha_k \in (0, \Lambda]$ with $\Lambda < \infty$ and*

$$\sum_{k\in\mathbb{N}} \alpha_k \max\{\|z_k - z_{k-1}\|_{M_k}, \|z_k - z_{k-1}\|^2_{M_k}\} < +\infty\,.$$

Suppose that $\operatorname{zer}(A+B) \neq \emptyset$. Then $(z_k)_{k\in\mathbb{N}}$ are bounded and weakly converge to a point $z^* \in zer(A+B)$, i.e. $z_k \rightharpoonup z^*$ as $k \to \infty$.

Furthermore, if additionally we assume $\epsilon_k \equiv 0$ for any $k \in \mathbb{N}$, $\gamma_A > 0$ or $\gamma_B > 0$ and $M_k - \frac{1}{\beta}I \in \mathcal{S}_\epsilon(\mathcal{H})$, then there exist some $\xi \in (0,1)$, some $\Theta > 0$ and some $K_0 \in \mathbb{N}$ such that for any $k \geq K_0$,

$$\|z_k - z^*\|^2_{M_k} \leq (1-\xi)^{k-K_0}\|z_{K_0} - z^*\|^2_{M_{K_0}} + \sum_{i=K_0}^{k-1} \Theta(1-\xi)^{k-i}\alpha_i \max\{\|z_i - z_{i-1}\|_{M_i}, \|z_i - z_{i-1}\|^2_{M_i}\}\,. \tag{17}$$

Proof. See Appendix A.3. ☐

Remark 1.
- *We obtain that the second term on the right hand of (17) converges to 0 by using [29, Lemma 2.2.3].*
- *The linear convergence factor $1-\xi$ is chosen such that there exists $K_0 \in \mathbb{N}$ with*

  $$(1+\eta_k)\Big(\frac{1-\frac{\gamma_B}{C}}{1+\frac{\gamma_A}{C}}\Big) \leq 1-\xi\,,$$

  for all $k \geq K_0$ holds.
- *The convergence rate for the strongly monotone setting can be influenced a lot by the decay rate of α_k.*
  - *(i) If $\alpha_k = O(q^k)$ for $q = 1-\xi$, we have convergence rate of $O(kq^k)$ for $k > K_0$ where K_0 is sufficiently large.*
  - *(ii) If $\alpha_k = O(\frac{1}{k^2})$, we have convergence rate of $O(\frac{1}{k})$ for $k > K_0$ where K_0 is sufficiently large.*

3.2.2 Algorithm 2: Quasi-Newton Forward-Backward Splitting with Relaxation

At the cost of a relaxation instead of an inertial step, we can significantly generalize the flexibility of choosing the metric. Note that without loss of generality, we assume $\tilde{z}_k \neq z_k$ for all $k \in \mathbb{N}$, since otherwise $\tilde{z}_k$ is already solves the inclusion problem after a finite number of iterations. This method is inspired by [17].

Assumption 2. *Let $\sigma \in (0, +\infty)$. $(M_k)_{k\in\mathbb{N}}$ is a sequence in $\mathcal{S}_\sigma(\mathcal{H})$ and $(\epsilon_k)_{k\in\mathbb{N}}$ is a sequence in $\mathcal{H}$ such that:*

(i) For all $k \in \mathbb{N}$, we have $(M_k - \frac{1}{\beta}I) \in \mathcal{S}_c(\mathcal{H})$, for some $c > 0$,

(ii) $\sup_{k\in\mathbb{N}} ||M_k|| < \infty$,

(iii) $\sum_{k\in\mathbb{N}} \|\epsilon_k\| < +\infty$.

Theorem 3.2. *Consider Problem (10) and let the sequence $(z_k)_{k\in\mathbb{N}}$ be generated by Algorithm 2 where Assumption 2 holds. Then $(\|z_k - z^*\|)_{k\in\mathbb{N}}$ is bounded for any $z^* \in zer(A+B)$ and $(z_k)_{k\in\mathbb{N}}$ weakly converges to some $z^* \in zer(A+B)$, i.e. $z_k \rightharpoonup z^*$ as $k \to \infty$.*

Moreover, if $\epsilon_k \equiv 0$, $\|z_k - z^*\|$ decreases for any $z^* \in zer(A+B)$ as $k \to \infty$. Furthermore, if $\gamma_A > 0$ or $\gamma_B > 0$ and $M_k - \frac{1}{\beta}I \in \mathcal{S}_\epsilon(\mathcal{H})$ for all $k \in \mathbb{N}$, then z_k converges linearly: there exist some $\xi \in (0,1)$ such that

$$\|z_k - z^*\|^2 \leq (1-\xi)^k \|z_0 - z^*\|^2 . \tag{18}$$

Proof. See Appendix A.4. □

Remark 2. *The linear convergence factor is given by*

$$\xi = \tfrac{1}{2}\min\{2(\gamma_A + \gamma_B)\delta, c\delta\} ,$$

where $\delta = \frac{c}{2(C+\frac{1}{\beta})^2}$.

Remark 3. *It is worth mentioning that Algorithm 2 can be interpreted as a closed loop system which uses the previous iterates to update the relaxation parameter α_k.*

4 Resolvent Calculus for Low-Rank Perturbed Metric

In this section, we extend the proximal calculus of [7] to the setting of resolvent operators J_T^V with a metric $V = M \pm Q \in \mathcal{S}_{++}(\mathcal{H})$, where $M \in \mathcal{S}_{++}(\mathcal{H})$ and $Q \in \mathcal{S}_0(\mathcal{H})$. This is a key result for making our quasi-Newton methods efficiently applicable for solving monotone inclusion problemss. Computing the resolvent operator $J_T^V(z)$ can be reduced to two simple problems consisting of evaluating J_T^M at a shifted point $z - M^{-1}v^* \in \mathcal{H}$ for some $v^* \in \mathcal{H}$ that is derived from an r-dimensional root finding problem which can be solved by a semi-smooth Newton method (Algorithm 3) or bisection method (Algorithm 4). In conclusion, if J_T^M can be computed efficiently, the same is true for J_T^V. The result crucially relies on the abstract duality principle of Attouch-Théra [4] (see Lemma 2.6).

4.1 General case

We first present a general result.

Theorem 4.1. *Let $T\colon \mathcal{H} \rightrightarrows \mathcal{H}$ be a maximally monotone operator, $V := M \pm Q$ be a symmetric positive definite metric with bounded linear operators $M, Q\colon \mathcal{H} \to \mathcal{H}$ such that M is positive definite and Q is positive semi-definite. Then, the resolvent operator J_T^V can be computed as follows:*

$$x^* = J_T^V(z) \quad \iff \quad \begin{cases} x^* = J_T^M(z - M^{-1}v^*) \text{ and} \\ v^* \text{ solves } 0 \in \pm Q^+ v^* + z - J_T^M(z - M^{-1}v^*) \\ \text{s.t. } v^* \in \operatorname{im}(\pm Q) . \end{cases} \tag{19}$$

The solution $v^* = U\alpha^*$ where α^* is the unique root of

$$l(\alpha) := U^* Q^+ U\alpha + U^*(z - J_T^M(z \mp M^{-1}U\alpha)) , \tag{20}$$

where Q^+ is the pseudo-inverse of Q and $U\colon \mathbb{R}^r \to \operatorname{im}(Q)$, $\alpha \mapsto U\alpha := \sum_{i=1}^r \alpha_i u_i$ is an isomorphism defined by r linearly independent $u_1, ..., u_r \in \operatorname{im}(Q)$. The function l is Lipschitz continuous with constant $||U^*Q^+U|| + ||M^{-1/2}U||^2$ and strictly monotone.

Now, we give the proof of Theorem 4.1. For convenience, we define translation operator $\tau_p\colon \mathcal{H} \to \mathcal{H}$ by $\tau_p(x) = x - p$ with inverse $\tau_p^{-1} = \tau_{-p}$.

Proof. See Appendix A.5. □

Corollary 4.1.1. *Consider the assumptions of Theorem 4.1 with Q given by $Q = UU^*$ for some $U \in \mathcal{B}(\mathbb{R}^r, \mathcal{H})$. In this case,*

$$x^* = J_T^V(z) \quad \iff \quad \begin{cases} x^* = J_T^M(z - M^{-1}U\alpha^*) \text{ and} \\ \alpha^* \text{ solves } l(\alpha) \end{cases} \tag{21}$$

where α^* is the unique root of

$$l(\alpha) := \alpha + \left\langle U, z - J_T^M(z \mp M^{-1}U\alpha) \right\rangle . \tag{22}$$

The function l is Lipschitz continuous with constant $1 + ||M^{-1/2}U||^2$ and strictly monotone.

Proof. See Appendix A.6. □

4.2 Solving the Root-Finding Problem

The efficiency of the reduction in Theorem 4.1 relies also on the solution of a root finding problem which we discuss thoroughly in this subsection. We present a semi-smooth Newton approach, for which we prove local convergence. In order to narrow down the neighborhood of the sought root, we complement the semi-smooth Newton strategy by a bisection method in Section 4.2.2.

4.2.1 Semi-smooth Newton Methods

In order to solve $l(\alpha) = 0$ in (20) efficiently, we turn to a semi-smooth Newton method. A locally Lipschitz function is called semi-smooth if its Clarke Jacobian defines a Newton approximation scheme [15, Definition 7.4.2]. If $l(\alpha)$ is semi-smooth and any element of the Clarke Jacobian $\partial^c l(\alpha^*)$ is non-singular, then we can apply the inexact semi-smooth Newton method detailed in Algorithm 3, analog to [7]. Semi-smoothness may seem very restrictive. However we will observe

---

Algorithm 3 Semi-smooth Newton method to solve $l(\alpha) = 0$

Require: A point $\alpha_0 \in \mathbb{R}^n$. N is the maximal number of iterations.
  Update for $k = 0, \cdots, N$:
  if $l(\alpha_k) = 0$ **then**
    stop
  else
    Select $G_k \in \partial^c l(\alpha_k)$, compute α_{k+1} such that

$$l(\alpha_k) + G_k(\alpha_{k+1} - \alpha_k) = e_k ,$$

  and $e_k \in \mathbb{R}^r$ is an error term satisfying $\|e_k\| \leq \eta_k \|G_k\|$ and $\eta_k \geq 0$.
  end if
  End

---

there exists a broad class of functions $l(\alpha)$ satisfying that $l(\alpha)$ is semi-smooth. As shown in [8], a tame locally Lipschitz function is semi-smooth. Therefore, it is sufficient to ensure $l(\alpha)$ is tame. We refer to [8] for the definition of tameness. If the monotone operator T in $l(\alpha)$ is a tame map, then $l(\alpha)$ is also tame. In this case, the convergence result for Algorithm 3 can be adapted from [7].

Proposition 4.2. *Let $l(\alpha)$ be defined as in Theorem 4.1, where T is a set-valued tame mapping. Then $l(\alpha)$ is semi-smooth and all elements of $\partial^c l(\alpha^*)$ are non-singular where α^* is the unique root of $l(\alpha)$ from (20). In turn there exists $\bar{\eta}$ such that if $\eta_k \leq \bar{\eta}$ for every k, there exists a neighborhood of α^* such that for all α_0 in that neighborhood, the sequence generated by Algorithm 3 is well defined and converges to α^* linearly. If $\eta_k \to 0$, the convergence is superlinear.*

Proof. See Appendix A.7. □

Example 1. *If f is a tame function and locally Lipschitz, then by [18, Proposition 3.1], ∂f is a tame map.*

Example 2. *The assumption that T is a tame mapping is not restrictive. For example, in PDHG setting we have a set-valued operator $T = \begin{pmatrix} \partial g & K^* \\ -K & \partial f \end{pmatrix}$ as defined by (33). If g and f are both tame functions, then ∂g and ∂f are tame as well [18]. As a result, T is a tame mapping.*

4.2.2 Bisection

We are going to solve a root finding problem for $l(\alpha) = 0$ via the bisection method in the Algorithm 4 in the special case when α is one-dimensional. We can still have similar bound on the range of α^* as in [7]. The key tool is a bound on the values of α given by the following proposition.

Proposition 4.3. *For $r = 1$, the root α^* of $l(\alpha) = 0$ lies in the set $[-\zeta, \zeta]$ where*

$$\zeta = \|u\|(2\|z\| + \|J_T^V(0)\|) . \tag{23}$$

Proof. See Appendix A.8. □

Algorithm 4 Bisection method to solve $l(\alpha) = 0$ when $r = 1$

Require: Tolerance $\epsilon \geq 0$, number of iterates N
  Compute the bound β from (23), and set $k = 0$.
  Set $\alpha_- = -\beta$ and $\alpha_+ = \beta$.
  Update for $k = 0, \cdots, N$:
  Set $\alpha_k = \frac{1}{2}(\alpha_- + \alpha_+)$.
  if $l(\alpha_k) > 0$ **then**
    $\alpha_+ \leftarrow \alpha_k$,
  else
    $\alpha_- \leftarrow \alpha_k$.
  end if
  if $k > 1$ and $|\alpha_k - \alpha_{k-1}| < \epsilon$ **then**
    return α_k
  end if
  End

Furthermore, we can combine the bisection method and the semi-smooth Newton method. Since the non-smooth Newton method is locally convergent, it requires a starting point in a sufficiently near neighborhood of the solution in order to guarantee convergence. By using bisection, we can obtain a sequence of points approaching the solution. When those points reach the neighborhood required for convergence of the semi-smooth Newton Method, we turn to use the semi-smooth Newton method to obtain faster convergence. In Proposition 4.2, α_0 is required to belong to a neighborhood of α^*, which can be achieved by Algorithm 4, i.e., we can assert to find a point α such that $|\alpha - \alpha^*| < \delta$ in $\log_2((2\zeta/\delta))$ steps, where ζ is as in (23).

4.3 Implementation of the quasi-Newton Forward-Backward Step

In this section, we consider a fixed iteration k and for convenience, we omit this index in this subsection. We consider low-rank perturbed metric $V = M \pm Q$ where $Q = UU^*$. Algorithm 1 and Algorithm 2 both have similar forward-backward steps. And those forward-backward steps require a kind of calculation in the form of $J_A^V(z - V^{-1}Bz)$ for some z which can be significantly simplified by applying Theorem 4.1. Instead of evaluating $J_A^V(z - V^{-1}Bz)$ directly, we split the whole task into two subproblems consisting of a root finding problem and an evaluation of resolvent with respect to M which is assumed to be simple to calculate. For simplicity, we denote $\check{z} = z - V^{-1}Bz$.

Proposition 4.4. *The forward-backward step from some $\check{z}$ to some $\hat{z}$ in Algorithm 1 and Algorithm 2 can be solved by*

$$\hat{z} = J_A^V(\check{z}) = J_A^M(\check{z} \mp M^{-1}U\alpha^*) . \tag{24}$$

Here, $\alpha^* \in \mathbb{R}^r$ is the unique root of $\mathcal{L} \colon \mathbb{R}^r \to \mathbb{R}^r$,

$$\mathcal{L}(\alpha^*) := U^*(\check{z} - J_A^M(\check{z} \mp M^{-1}U\alpha^*)) + \alpha^* = 0 . \tag{25}$$

We notice that the calculation of V^{-1} is expensive when the size of V is large. We can use Sherman-Morrison formula. However the inverse of M is inevitable and in algorithms that we propose in Section 6 it can be computationally hard. So, we introduce Proposition 4.5 to avoid computing M_k^{-1}.

Proposition 4.5. *The forward-backward step from some $\check{z}$ to some $\hat{z}$ in Algorithm 1 and Algorithm 2 can be solved by*

$$\hat{z} = J_A^V(\check{z}) = J_A^M(z - M^{-1}Bz \mp M^{-1}U\xi^*). \tag{26}$$

Here, $\xi^* \in \mathbb{R}^r$ is the unique zero of $\mathcal{J}\colon \mathbb{R}^r \to \mathbb{R}^r$,

$$\mathcal{J}(\xi^*) \coloneqq U_k^*(z_k - J_A^M(z - M^{-1}Bz \mp M^{-1}U\xi^*)) + \xi^* = 0\,. \tag{27}$$

The function $\mathcal{J}$ is Lipschitz continuous with constant $1 + ||M^{-1/2}U||^2$ and strictly monotone.

Proof. See Appendix A.9. □

5 A General 0SR1 Inertial Quasi-Newton Method for Monotone Inclusion

5.1 General algorithm with generic metric $M \pm Q_k$

We note that if we set in the inclusion problem (10) $A \equiv 0$, and $B = \nabla f$ of some convex smooth function f, the update step consisting of (12) and (11) reduces to Gradient Descent when $M_k \equiv I$ and to the classic Newton method when $M_k = \nabla^2 f(z_k)$. Motivated by classic Newton and quasi-Newton methods, we construct M_k as an approximation of the differential of Bz_k at z_k. We generalize the quasi-Newton method 0SR1 from differentiable ∇f (SR1 method with 0-memory) to a cocoercive operator B. The approximation M_k shall satisfy the modified secant condition:

$$M_k s_k = y_k, \quad \text{where} \quad y_k = Bz_k - Bz_{k-1}, \quad s_k = z_k - z_{k-1}\,. \tag{28}$$

Choose $M_0 \in \mathcal{S}_{++}(\mathcal{H})$ which is positive definite. The update is:

$$M_k = M_0 \pm \gamma_k u_k u_k^*, \quad u_k = (y_k - M_0 s_k)/\sqrt{|\langle y_k - M_0 s_k, s_k\rangle|}, \tag{29}$$

where $\gamma_k \in [0, +\infty)$ needs to be selected such that M_k is positive definite and $\pm$ depends on the sign of $\langle y_k - M_0 s_k, s_k\rangle$. If $\langle y_k - M_0 s_k, s_k\rangle > 0$, we use

$$M_k = M_0 + \gamma_k u_k u_k^*, \tag{30}$$

and, if $\langle y_k - M_0 s_k, s_k\rangle < 0$, we use

$$M_k = M_0 - \gamma_k u_k u_k^*. \tag{31}$$

In the rest part of this section, we investigate the conditions for M_k defined above to meet Assumption 1 and Assumption 2.

Lemma 5.1. *Let M_0 be symmetric positive definite, $\frac{1}{\beta}$ be Lipschitz constant of operator B.*

(i) If $M_k = M_0 + \gamma_k u_k u_k^*$ and $\gamma_k > 0$, $\forall k \in \mathbb{N}$, M_k is always positive definite and symmetric and Assumption 2 is guaranteed.

(ii) If $M_k = M_0 - \gamma_k u_k u_k^*$ and $\gamma_k \le c\frac{\rho_{min}(M_0 - \frac{1}{\beta}I)}{\|u_k\|^2}$ where $\rho_{min}(M_0 - \frac{1}{\beta}I) > 0$ is the smallest eigenvalue of matrix $M_0 - \frac{1}{\beta}$ and $c \in (0, 1)$, then M_k is positive definite and Assumption 2 is satisfied.

Now, in order to guarantee Assumption 1 for $M_k \coloneqq M_0 \pm Q_k$ with $Q_k = \gamma_k u_k u_k^*$, we introduce the following lemma.

Lemma 5.2. *Let $(\eta_k)_k \in \ell_+^1(\mathbb{N})$, $M_0 - \frac{1}{\beta}I$ be positive definite, $M_k \coloneqq M_0 \pm \gamma_k u_k u_k^*$ and $\gamma_k = \frac{\eta_k}{\|u_k\|_2^2}$ with $\eta_k \le \rho_{min}(M_0 - \frac{1}{\beta}I)$ for any $k \in \mathbb{N}$. Then, Assumption 1 is satisfied.*

6 Inertial Quasi-Newton PDHG for Saddle-point Problems

In this section, we consider a min-max problem as follows:

$$\min_{x\in\mathcal{H}_1}\max_{y\in\mathcal{H}_2} g(x) + G(x) + \langle Kx, y\rangle - f(y) - F(y) \tag{32}$$

with a linear mapping $K\colon \mathcal{H}_1 \to \mathcal{H}_2$, proper lower semi-continuous convex functions $g\colon \mathcal{H}_1 \to \overline{\mathbb{R}} := \mathbb{R}\cup\{\pm\infty\}$ and $f\colon \mathcal{H}_2 \to \overline{\mathbb{R}}$, and continuously differentiable convex functions $G\colon \mathcal{H}_1 \to \mathbb{R}$ and $F\colon \mathcal{H}_2 \to \mathbb{R}$ with Lipschitz continuous gradient. This problem can be expressed as a special monotone inclusion problem from which we derive in Algorithm 5 an inertial quasi-Newton Primal-Dual Hybrid Gradient Method (PDHG) and in Algorithm 6 a quasi-Newton PDHG with relaxation step as special applications of Algorithm 1 and Algorithm 2 respectively (see Proposition 6.1). By Fermat's rule, the optimality condition for (32) is the following inclusion problem:

$$0 \in T(z) + B(z) \quad \text{with} \quad z = \begin{pmatrix} x \\ y\end{pmatrix}, \quad \text{where } T(z) = \begin{pmatrix} \partial g(x) + K^*y \\ -Kx + \partial f(y)\end{pmatrix} \text{ and } B(x) = \begin{pmatrix} \nabla G(x) \\ \nabla F(y)\end{pmatrix}. \tag{33}$$

Algorithm 5 Inertial quasi-Newton PDHG

Require: $N \geq 0$, $(\|\epsilon_k\|)_{k\in\mathbb{N}} \in \ell^1_+(\mathbb{N})$

Update for $k = 0, \cdots, N$:

(i) Compute M_k according to a quasi-Newton framework.

(ii) Compute the inertial step with parameter α_k:

$$\begin{aligned} \bar{x}_k &= x_k + \alpha_k(x_k - x_{k-1}), \\ \bar{y}_k &= y_k + \alpha_k(y_k - y_{k-1}). \end{aligned} \tag{34}$$

(iii) Compute the main quasi-Newton PDHG step:

$$\begin{aligned} x_{k+1} &= \operatorname{prox}_g^{\mathcal{T}}(\bar{x}_k - \mathcal{T}\nabla G(\bar{x}_k) - \mathcal{T}K^*\bar{y}_k \mp \mathcal{T}U_{k,x}\xi_k) + \epsilon_{k,x}, \\ y_{k+1} &= \operatorname{prox}_f^{\Sigma}(\bar{y}_k - \Sigma\nabla F(\bar{y}_k) + \Sigma K(2x_{k+1} - \bar{x}_k) \mp \Sigma U_{k,y}\xi_k) + \epsilon_{k,y}, \end{aligned} \tag{35}$$

where ξ_k solves $\mathcal{J}(\xi_k) = 0$ and $\epsilon_k = \begin{pmatrix} \epsilon_{k,x} \\ \epsilon_{k,y}\end{pmatrix}$ is the error caused by computation at the k-th iterate.

End

Proposition 6.1 (PDHG method as a special proximal point algorithm [17]). *The update step of PDHG can be regarded as a proximal point algorithm with special metric M:*

$$M(z_{k+1} - z_k) + T(z_{k+1}) \ni -B(z_k) \quad \textit{where} \quad M = \begin{pmatrix} \mathcal{T}^{-1} & -K^* \\ -K & \Sigma^{-1}\end{pmatrix}. \tag{40}$$

Therefore, we are able to apply Algorithm 1 and Algorithm 2 directly to a saddle point problem by regarding T as A in the inclusion problem (10), which will give us Algorithm 5 and 6, respectively. We are going to use a variable metric $M_k = M \pm Q_k \in \mathcal{S}(\mathcal{H}_1 \times \mathcal{H}_2)$ instead of $M \in \mathcal{S}(\mathcal{H}_1 \times \mathcal{H}_2)$ in (40) for the k-th iterate. Here, we set $Q_k = U_kU_k^*$ where $U_k\colon \mathbb{R}^r \to \operatorname{im}(Q)$, $\alpha \mapsto U_k(\alpha) := \sum_{i=1}^r \alpha_i u_{k,i}$ is an isomorphism defined by r linearly independent $u_{k,1}, ..., u_{k,r} \in \operatorname{im}(Q)$ for each k-th iterate. We obtain the update step: Find $z_{k+1} \in \mathcal{H}_1 \times \mathcal{H}_2$ such that

$$M_k(z_{k+1} - z_k) + T(z_{k+1}) \ni -B(z_k). \tag{41}$$

In practice, Proposition 4.5 which is derived from our main result Theorem 4.1 turns out to be more tractable. To show how to calculate the update step, using Proposition 4.5 in quasi-Newton PDHG in Algorithm 5 and in Algorithm 6, we introduce Proposition 6.2. Let $U_k \in \mathcal{B}(\mathbb{R}^r, \mathcal{H}_1 \times \mathcal{H}_2)$. We set $U_k = \begin{pmatrix} U_{k,x} \\ U_{k,y}\end{pmatrix}$ with $U_{k,x} \in \mathcal{B}(\mathbb{R}^r, \mathcal{H}_1)$ and $U_{k,y} \in \mathcal{B}(\mathbb{R}^r, \mathcal{H}_2)$. The validity of the update step is verified in Proposition 6.2.

Algorithm 6 Quasi-Newton PDHG with relaxation

Require: $N \geq 0$, $(\|\epsilon_k\|)_{k\in\mathbb{N}} \in \ell^1_+(\mathbb{N})$

Update for $k = 0, \cdots, N$:

(i) Compute M_k according to quasi-Newton framework.

(ii) Compute the main quasi-Newton PDHG step:

$$\begin{aligned} \tilde{x}_k &= \operatorname{prox}_g^{\mathcal{T}}(x_k - \mathcal{T}\nabla G(x_k) - \mathcal{T}K^* y_k \mp \mathcal{T}U_{k,x}\xi_k) + \epsilon_{k,x}\,, \\ \tilde{y}_k &= \operatorname{prox}_f^{\Sigma}(y_k - \Sigma\nabla F(y_k) + \Sigma K(2\tilde{x}_k - x_k) \mp \Sigma U_{k,y}\xi_k) + \epsilon_{k,y}\,, \end{aligned} \tag{36}$$

where ξ_k solves $\mathcal{J}(\xi_k) = 0$ and $\epsilon_k = \begin{pmatrix}\epsilon_{k,x}\\ \epsilon_{k,y}\end{pmatrix}$ is the error caused by computation at the k-th iterate.

(iii) Relaxation step:

$$v_k \coloneqq M_k\left(\begin{pmatrix}x_k\\ y_k\end{pmatrix} - \begin{pmatrix}\tilde{x}_k\\ \tilde{y}_k\end{pmatrix}\right) + \begin{pmatrix}\nabla G(\tilde{x}_k)\\ \nabla F(\tilde{y}_k)\end{pmatrix} - \begin{pmatrix}\nabla G(x_k)\\ \nabla F(y_k)\end{pmatrix} \tag{37}$$

to compute the relaxation parameter t_k

$$t_k = \frac{\left\langle \begin{pmatrix}x_k\\ y_k\end{pmatrix} - \begin{pmatrix}\tilde{x}_k\\ \tilde{y}_k\end{pmatrix}, v_k \right\rangle}{2\|v_k\|^2} \tag{38}$$

and update x_k and y_k as follows

$$\begin{pmatrix}x_{k+1}\\ y_{k+1}\end{pmatrix} \leftarrow \begin{pmatrix}x_k\\ y_k\end{pmatrix} - t_k v_k\,. \tag{39}$$

End

Proposition 6.2. *The update step from $\bar{z}_k$ (z_k) to z_{k+1} ($\tilde{z}_k$) in Algorithm 5 (Algorithm 6) reduces for the quasi-Newton PDHG update step to compute*

$$\begin{cases} x_{k+1} &= \operatorname{prox}_g^{\mathcal{T}}(\bar{x}_k - \mathcal{T}\nabla G(\bar{x}_k) - \mathcal{T}K^*\bar{y}_k \mp \mathcal{T}U_{k,x}\xi_k) \\ y_{k+1} &= \operatorname{prox}_f^{\Sigma}(\bar{y}_k - \Sigma\nabla F(\bar{y}_k) + \Sigma K(2x_{k+1} - \bar{x}_k) \mp \Sigma U_{k,y}\xi_k)\,. \end{cases} \tag{42}$$

where, $\xi_k \in \mathbb{R}^r$ is the unique zero of $\mathcal{J}\colon \mathbb{R}^r \to \mathbb{R}^r$:

$$\begin{aligned} \mathcal{J}(\xi) = (U_{k,x})^*[&\underbrace{\operatorname{prox}_g^{\mathcal{T}}(\bar{x}_k - \mathcal{T}\nabla G(\bar{x}_k) - \mathcal{T}K^*\bar{y}_k \mp \mathcal{T}U_{k,x}\xi)}_{x_{k+1}(\xi)} - \bar{x}_k] \\ &+ (U_{k,y})^*[\operatorname{prox}_f^{\Sigma}(\bar{y}_k - \Sigma\nabla F(\bar{y}_k) + \Sigma K(2x_{k+1}(\xi) - \bar{x}_k) \mp \Sigma U_{k,y}\xi) - \bar{y}_k] - \xi\,. \end{aligned} \tag{43}$$

Remark 4. *By switching $\bar{z}_k$ with z_k and z_{k+1} with $\tilde{z}_k$, we obtain the update step for Algorithm 6.*

We would like to emphasize that using Proposition 6.2, we can avoid the computation of M^{-1} in the primal and dual setting, which is a computationally significant advantage. The convergence of Algorithms 5 and 6 is a direct consequence of Theorems 3.1 and 3.2.

Proposition 6.3 (Convergence of quasi-Newton PDHG method)**.** *Let $M_0 = M$ as defined in (40) and $M_k = M_0 \pm U_k U_k^*$.*

(i) If $(M_k)_{k\in\mathbb{N}}$ satisfies Assumption 1, then (x_k, y_k) generated by Algorithm 5 converges weakly to some solution of (2). Furthermore, if g and f are both strongly convex (or G and F are both strongly convex), then we obtain the same convergence rate as in Theorem 3.1.

(ii) If $(M_k)_{k\in\mathbb{N}}$ satisfies Assumption 2. Thus, (x_k, y_k) generated by Algorithm 6 converges weakly to some solution of (2). Furthermore, if g and f are both strongly convex (or G and F are both strongly convex) , then we obtain linear convergence.

7 Numerical experiments

The algorithms that we analyze in the experiment are summarized in Table 1.

Algorithm	Algorithm Name	metric
Forward-Backward Splitting (FBS)	Foward-Backward Primal-Dual Hybrid Gradient Method	M fixed as in (40)
Inertial Forward-Backward Splitting (IFBS)	Inertial Primal-Dual Hybrid Gradient Method	M fixed as in (40)
quasi-Newton Forward-Backward Splitting (QN-FBS)	quasi-Newton Primal-Dual Hybrid Gradient Method Gradient	Variable metric as in (29) with $M_0 = M$ from (40)
Relaxation quasi-Newton Forward-Backward Splitting (RQN-FBS)	Primal-Dual Hybrid Gradient Method with relaxation step	Variable metric as in (29) with $M_0 = M$ from (40)
Inertial quasi-Newton Forward-Backward Splitting (IQN-FBS)	Inertial quasi-Newton Primal-Dual Hybrid Gradient Method	Variable metric as in (29) with $M_0 = M$ from (40)

Table 1: Summary of algorithms used in the numerical experiments. Details are provided within each section.

Note that PDHG is used interchangeably as FBS in the later experiments since PDHG is a specialization of FBS.

7.1 TV-l_2 deconvolution

In this experiment, we solve a problem that is used for an image deconvolution [9]. Given a blurry and noisy image $b \in \mathbb{R}^{MN}$ (interpreted as a vector by stacking the M column vectors of length N), we seek to find a clean image $x \in \mathbb{R}^{MN}$ by solving the following optimization problem:

$$\min_{0 \le x \le 255} \frac{1}{2}\|Ax - b\|_2^2 + \mu\|Dx\|_{2,1}\,, \tag{44}$$

where $A \in \mathbb{R}^{MN \times MN}$ is a linear operator that acts as a blurring operator and $\|Dx\|_{2,1}$ implements a discrete version of the isotropic total variation norm of x using simple forward differences in horizontal and vertical direction. The parameter $\mu > 0$ stresses as the influence of the regularization term $\|Dx\|_{2,1}$ vs the data fidelity term $\frac{1}{2}\|Ax - b\|^2$. In order to deal with the non-smoothness, we rewrite the problem as a saddle point problem.

$$\min_x \max_y \langle Dx, y\rangle + \delta_\Delta(x) + \frac{1}{2}\|Ax - b\|_2^2 - \delta_{\{\|\cdot\|_{2,\infty} \le \mu\}}(y)\,, \tag{45}$$

where $\Delta := \{x \in \mathbb{R}^{MN} | 0 \le x_i \le 255, \forall i\}$. We can cast this problem into the general class of problems (32) by setting $K = D$, $f = \delta_{\{\|\cdot\|_{2,\infty} \le \lambda\}}$, $G(x) = \frac{1}{2}\|Ax - b\|_2^2$, $F(p) = 0$ and $g = \delta_V$. Here, G is L-smooth with $L = 1$ since we assume $\rho_{max}(A^*A) \le 1$, which is the largest eigenvalue of A^*A. Let $z_k = \begin{pmatrix} x_k \\ y_k \end{pmatrix}$. We compute the low-rank part $Q_k = \gamma_k u_k u_k^*$ by (29) with $Bz_k = \begin{pmatrix} A^*Ax_k - A^*b \\ 0 \end{pmatrix}$, which leads to a metric that just affects the primal update. In each iteration, we use the bisection Algorithm 4 and the semi-smooth Newton method 3 to locate the root.

Figure 1 shows the primal gap where the optimal primal value was computed by running the original PDHG method for 10000 iterates. For the variable metric at iterate k, we fixed $\gamma_k = 5/\|u_k\|_2^2$. Thus, the Assumption 2 is satisfied and the convergence of RQN-FBS (Algorithm 2) is guaranteed. However, with the same setting, Assumption 1 is not satisfied since there is no sequence $(\eta_k)_k \in \ell_+^1$ such that $M_{k+1} \preceq (1 + \eta_k)M_k$ for any $k \in \mathbb{N}$. In this practical problem, we still observe the convergence of IQN-FBS (Algorithm 1). We notice that our quasi-Newton type algorithms IQN-FBS, RQN-FBS and QN-FBS are much faster than original FBS algorithm and inertial FBS (IFBS). This can be explained by the fact that the Hessian of $G(x)$ is not identity and then by applying our quasi-Newton SR1 methods, we can adapt metric to the local geometry of the objective. All three quasi-Newton type algorithms share similar convergence rates in the end. The extrapolation can not speed up those algorithms at the beginning as shown by IFBS and IQN-FBS since for first several iterates $\|z_k - z_{k-1}\|$ are large and extrapolation parameter is small. However it does help eventually.

7.2 TV-l_2 deconvolution with infimal convolution type regularization

A source of optimization problems that fits (32) is derived from the following:

$$\min_{x \in \mathbb{R}^n} g(x) + G(x) + (f \Box h)(Dx), \tag{46}$$

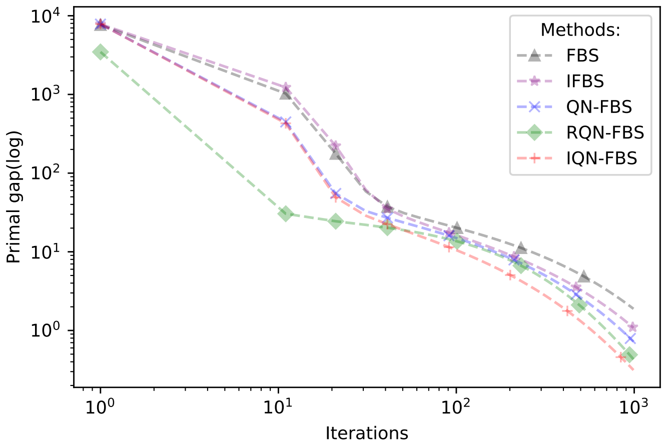

Figure 1: We compare convergence of the inertial quasi-Newton PDHG (IQN-FBS) to other algorithms in the Table 1 with $\mu = 0.001$, $\tau = 0.09$, $\sigma = 0.9$ and inertial parameter $\alpha_0 = 10$, $\alpha_k = \frac{10}{k^{1.1}(\max\{\|z_k - z_{k-1}\|, \|z_k - z_{k-1}\|^2\})}$. We also observe that the curve of RQN-FBS converges faster at the beginning. Our three quasi-Newton type algorithm RQN-, QN-, and IQN-FBS clearly outperform the original FBS and IFBS algorithm.

where $f \square h(\cdot) := \inf_{v\in\mathbb{R}^m} f(v) + h(\cdot - v)$ denotes the infimal convolution of f and h. As a prototypical image processing problem, we define a regularization term as infimal convolution between the total variation norm and a weighted squared norm, i.e. $g = 0$, $G(x) = \frac{1}{2}\|Ax - b\|^2$, $h(\cdot) = \frac{1}{2}\|W\cdot\|^2$ and $f(\cdot) = \mu\|\cdot\|_{2,1}$. This yields the problem:

$$\min_x \frac{1}{2}\|Ax - b\|^2 + \mu R(Dx) \tag{47}$$

where $R(\cdot) := \inf_{v\in\mathbb{R}^m} \|v\|_{2,1} + \frac{1}{2\mu}\|W(\cdot - v)\|^2$, W is a diagonal matrix of weights which is given to favor discontinuities along image edges and A, b, D are defined as in the first experiment. In practice, W can be computed by additional edge finding steps or by extra information. Here, we select W such that $\frac{1}{2} \le \|W\| \le 1$. The optimization problem (47) given in primal type can be converted into the saddle point problem:

$$\min_x \max_y \langle Dx, y\rangle + \frac{1}{2}\|Ax - b\|^2 - \delta_{\{\|\cdot\|_{2,+\infty}\le\mu\}}(y) - \frac{1}{2}\|W^{-1}y\|^2 \,. \tag{48}$$

We compute the low-rank part Q_k by (29) with $Bz_k = \begin{pmatrix} A^*Ax_k - A^*b \\ (W^{-1})^*W^{-1}y_k \end{pmatrix}$ which leads to metric that affects both primal and dual update. Here, $\beta = 1/4$. In each iteration, we combine the bisection (Algorithm 4) and the semi-smooth Newton method (Algorithm 3) to locate the root.

Figure 2 also shows the primal gap where the optimal primal value was still computed by running original PDHG for 10000 iterates. For the variable metric at iterate k, we fixed $\gamma_k = 2/\|u_k\|^2$. Assumption 2 is always satisfied. But Assumption 1 is not satisfied. We can observe from Figure 2: though RQN-FBS turned out to be the slowest one for this experiment, IQN-FBS is still the fast one. We observe that two quasi-Newton type methods (IQN-FBS and QN-FBS) converge faster than IFBS and FBS. Inertial methods (IQN-FBS, IFBS) are slightly faster respectively than QN-FBS and FBS.

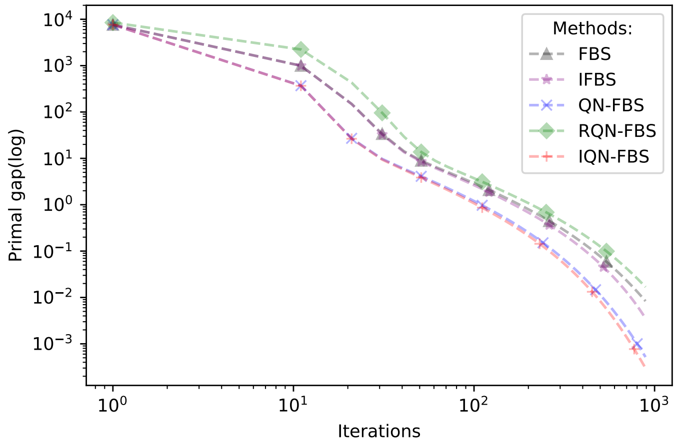

Figure 2: We compare convergence of the inertial quasi-Newton PDHG (IQN-FBS) to other algorithms in the Table 1 with $\mu = 0.01$, $\tau = 0.1$, $\sigma = 0.1$ and the extrapolation parameter $\alpha_0 = 1$, $\alpha_k = \max\{\frac{10}{k^{1.1}(\max\{\|z_k - z_{k-1}\|, \|z_k - z_{k-1}\|^2\})}, 1\}$. Our quasi-Newton type algorithm IQN-FBS can converge faster than the original FBS and IFBS algorithm.

7.2.1 Image denoising

Let $A = I$ in (48). We then obtain an image denoising problem with special norm defined by infimal convolution of total variation and weighted norm, which has strong convexity for both primal part and dual part. Besides, due to the simple formula, we obtain dual problem explicitly which means we can calculate primal and dual gap. The dual problem reads:

$$\max_{\|y\|_{2,\infty} \leq 1} -\frac{1}{2}\|K^* y - b\|^2 - \frac{1}{2}\|W^{-1} y\|^2 . \tag{49}$$

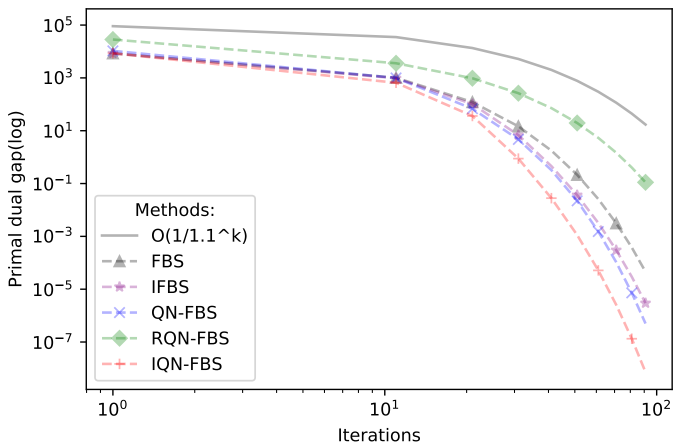

Figure 3: We compare convergence of the inertial quasi-Newton PDHG (IQN-FBS) to other algorithms in the Table 1 with $\mu = 0.1$, $\tau = 0.1$, $\sigma = 0.1$ and extrapolation parameter $\alpha_0 = 10$, $\alpha_k = \frac{10}{\max\{k^2, k^2\|z_k - z_{k-1}\|^2\}}$. All algorithms converge linearly and faster than $O(\frac{1}{1.1^k})$. Our quasi-Newton type algorithm IQN-FBS can converge faster than the original FBS and IFBS algorithm.

Figure 3 shows the primal dual gap and it will decrease to zero by using any algorithm from Table 1. To construct Q_k, we use $\gamma_k = \frac{2}{\|u_k\|^2}$. As we can observe from Figure 3, we have linear convergence for quasi-Newton type methods as what we expected in Theorem 3.1 and 3.2.

7.3 Conclusion

In this paper, we extended the framework of [7] for variable metrics to the setting of resolvent operators, solving efficiently the monotone inclusion problem (10) consisting of a set-valued operator A and a single-valued operator B. We proposed two variants of quasi-Newton Forward-Backward Splitting. We develop a general efficient Resolvent calculus that perfectly applies to this quasi-Newton setting. The convergence of the variant with relaxation requires mild assumptions on the metric which are easier to satisfy, whereas the other variant implements an inertial feature and is therefore often fast. As a special case of this framework, we developed an inertial quasi-Newton primal-dual algorithm that can be flexibly applied to a large class of saddle point problems.

Appendices

A Appendix

A.1 Proof of Lemma 2.3

Proof. We assume $y = M^{-1/2} \circ J_{M^{-1/2}TM^{-1/2}} \circ M^{1/2}(z)$. Then, we obtain

$$\begin{aligned} y &= M^{-1/2} \circ J_{M^{-1/2}TM^{-1/2}} \circ M^{1/2}(z) \\ M^{1/2}y &= J_{M^{-1/2}TM^{-1/2}} \circ M^{1/2}(z) \\ M^{1/2}y &= (I + M^{-1/2}TM^{-1/2})^{-1}(M^{1/2}z) \\ (M^{1/2}z) &\in (I + M^{-1/2}TM^{-1/2})M^{1/2}y \\ (M^{1/2}z) &\in (M^{1/2} + M^{-1/2}T)y \\ Mz &\in (M+T)y \\ y &= (M+T)^{-1}(Mz) \\ y &= J_T^M(z)\,. \end{aligned} \tag{50}$$

□

A.2 Proof of Lemma 2.4

Proof. This proof is adapted from [5]. Let $(u, v) = (J_A^M(x), J_A^M(y))$ for some $x, y \in \mathcal{H}$. By the definition of resolvent operator J_A^M, we obtain

$$u = J_A^M(x) \iff M(x-u) \in Au\,.$$

Similarly, we obtain $M(y - v) \in Av$. Then γ_A-strong monotonicity of A yields

$$\begin{aligned} \langle M(x-u) - M(y-v), u-v\rangle &\geq \gamma_A \|u-v\|^2 \\ \langle M(x-y), u-v\rangle - \langle M(u-v), u-v\rangle &\geq \gamma_A \|u-v\|^2 \\ \langle M(x-y), u-v\rangle &\geq \gamma_A \|u-v\|^2 + \|u-v\|_M^2\,. \end{aligned}$$

Since $\|M\|$ is bounded by C, we obtain

$$\langle M(x-y), u-v\rangle \geq \gamma_A \|u-v\|^2 + \|u-v\|_M^2 \geq (1 + \tfrac{\gamma_A}{C})\|u-v\|_M^2\,. \tag{51}$$

Consequently, J_A^M is $(1 + \frac{\gamma_A}{C})$-cocoercive and Lipschitz continuous with constant $1/(1 + \frac{\gamma_A}{C})$ with respect to the norm $\|\cdot\|_M$. □

A.3 Proof of Theorem 3.1

Proof. For convenience, we set $B_k := M_k^{-1}B$. Fix $z \in zer(A+B)$ which is equivalent to $z = J_A^{M_k}(z - B_k z)$. We set $\hat{z}_{k+1} := J_A^{M_k}(\bar{z}_k - B_k \bar{z}_k)$, i.e., $z_{k+1} = \hat{z}_{k+1} + \epsilon_k$.
Boundedness:

First, we are going to show that $\|z_k - z\|_{M_k}$ is bounded. The following are several useful estimations we will use later. Assumption 1 yields

$$\|z_{k+1} - z\|^2_{M_{k+1}} \leq (1+\eta_k)\|z_{k+1} - z\|^2_{M_k}\,. \tag{52}$$

Since $z_{k+1} = \hat{z}_{k+1} + \epsilon_k$, it follows that

$$\|z_{k+1} - z\|^2_{M_k} = \|\hat{z}_{k+1} - z + \epsilon_k\|^2_{M_k} \leq \|\hat{z}_{k+1} - z\|^2_{M_k} + 2\|\epsilon_k\|_{M_k}\|\hat{z}_{k+1} - z\|_{M_k} + \|\epsilon_k\|^2_{M_k}\,. \tag{53}$$

The assumption that B is β-cocoercive yields that

$$\langle \bar{z}_k - z, B_k\bar{z}_k - B_k z\rangle_{M_k} = \langle \bar{z}_k - z, B\bar{z}_k - Bz\rangle \geq \beta\|B\bar{z}_k - Bz\|^2\,. \tag{54}$$

The assumption that $M_k - \frac{1}{2\beta}I \in \mathcal{S}_\epsilon(\mathcal{H})$ yields that $2\beta I - M_k^{-1} \in \mathcal{S}_{++}(\mathcal{H})$.
The cocoercivity of B and monotonicity of B yield the following estimations (I) and (II) respectively:

$$\begin{aligned}
\|(\bar{z}_k - B_k\bar{z}_k) - (z - B_k z)\|^2_{M_k} &= \|\bar{z}_k - z\|^2_{M_k} - 2\langle \bar{z}_k - z, B_k\bar{z}_k - B_k z\rangle_{M_k} + \|B_k\bar{z}_k - B_k z\|^2_{M_k}\\
&\overset{\text{(i)}}{=} \|\bar{z}_k - z\|^2_{M_k} - 2\langle \bar{z}_k - z, B\bar{z}_k - Bz\rangle + \|B\bar{z}_k - Bz\|^2_{M_k^{-1}}\\
&\overset{\text{(ii)}}{\leq} \|\bar{z}_k - z\|^2_{M_k} - \|B\bar{z}_k - Bz\|^2_{2\beta - M_k^{-1}} \qquad \text{(I)}\\
&\overset{\text{(iii)}}{\leq} \|\bar{z}_k - z\|^2_{M_k} - \|B\bar{z}_k - Bz\|^2_{\beta - M_k^{-1}} - \gamma_B\|\bar{z}_k - z\|^2\,, \qquad \text{(II)}
\end{aligned}$$

where (i) uses $\|B_k\bar{z}_k - B_k z\|^2_{M_k} = \|B\bar{z}_k - Bz\|^2_{M_k^{-1}}$, (ii) uses (54) and (iii) uses monotonicity of B. Note that we use the shorthand $\beta - M_k^{-1}$ for $\beta I - M_k^{-1}$. The fact that $J_A^{M_k}$ is firmly non-expansive since A is maximally monotone with respect to M_k implies that

$$\begin{aligned}
\|\hat{z}_{k+1} - z\|^2_{M_k} &= \|J_A^{M_k}(\bar{z}_k - B_k\bar{z}_k) - J_A^{M_k}(z - B_k z)\|^2_{M_k}\\
&\leq \|(\bar{z}_k - B_k\bar{z}_k) - (z - B_k z)\|^2_{M_k}\\
&\quad - \|(I - J_A^{M_k})(\bar{z}_k - B_k\bar{z}_k) - (I - J_A^{M_k})(z - B_k z)\|^2_{M_k}\\
&\leq \|\bar{z}_k - z\|^2_{M_k} - \|B\bar{z}_k - Bz\|^2_{2\beta - M_k^{-1}} - \|(\bar{z}_k - \hat{z}_{k+1}) - (B_k\bar{z}_k - B_k z)\|^2_{M_k}\,,
\end{aligned} \tag{55}$$

where the last inequality uses (I). It follows from Assumption 1 that the term $\|B\bar{z}_k - Bz\|^2_{2\beta - M_k^{-1}} \geq 0$. We continue to bound the first term on the right hand side of (55). Using [5, Lemma 2.14] and the definition of $\bar{z}_k$, we obtain the following:

$$\|\bar{z}_k - z\|^2_{M_k} = (1+\alpha_k)\|z_k - z\|^2_{M_k} - \alpha_k\|z_{k-1} - z\|^2_{M_k} + (1+\alpha_k)\alpha_k\|z_k - z_{k-1}\|^2_{M_k}\,, \tag{56}$$

and by using the triangle inequality, we also obtain another estimation:

$$\|\bar{z}_k - z\|_{M_k} \leq (1+\alpha_k)\|z_k - z\|_{M_k} + \alpha_k\|z_{k-1} - z\|_{M_k}\,. \tag{57}$$

In order to address complete update step, we make the following estimation:

$$
\begin{aligned}
\|z_{k+1}-z\|^2_{M_{k+1}} &\le (1+\eta_k)\|z_{k+1}-z\|^2_{M_k} \\
&\overset{\text{(i)}}{\le} (1+\eta_k)\big(\|\hat{z}_{k+1}-z\|^2_{M_k} + 2\|\epsilon_k\|_{M_k}\|\hat{z}_{k+1}-z\|_{M_k} + \|\epsilon_k\|^2_{M_k}\big) \\
&\overset{\text{(ii)}}{\le} (1+\eta_k)\big(\|\bar{z}_k-z\|^2_{M_k} + 2\|\epsilon_k\|_{M_k}\|\bar{z}_k-z\|_{M_k} + \|\epsilon_k\|^2_{M_k} \\
&\quad - \|B\bar{z}_k - Bz\|^2_{2\beta-M_k^{-1}} - \|(\bar{z}_k-\hat{z}_{k+1}) - (B_k\bar{z}_k - B_kz)\|^2_{M_k}\big) \\
&\overset{\text{(iii)}}{\le} (1+\eta_k)\big((1+\alpha_k)\|z_k-z\|^2_{M_k} - \alpha_k\|z_{k-1}-z\|^2_{M_k} \\
&\quad + (1+\alpha_k)\alpha_k\|z_k-z_{k-1}\|^2_{M_k} \\
&\quad + 2\|\epsilon_k\|_{M_k}((1+\alpha_k)\|z_k-z\|_{M_k} + \alpha_k\|z_{k-1}-z\|_{M_k}) \\
&\quad + \|\epsilon_k\|^2_{M_k} - \|B\bar{z}_k - Bz\|^2_{2\beta-M_k^{-1}} - \|(\bar{z}_k-\hat{z}_{k+1}) - (B_k\bar{z}_k - B_kz)\|^2_{M_k}\big) \\
&\overset{\text{(iv)}}{\le} (1+\eta_k)\big(\|z_k-z\|^2_{M_k} \\
&\quad + \alpha_k(\|z_k-z\|_{M_k} - \|z_{k-1}-z\|_{M_k})(\|z_k-z\|_{M_k} + \|z_{k-1}-z\|_{M_k}) \\
&\quad + (1+\alpha_k)\alpha_k\|z_k-z_{k-1}\|^2_{M_k} \\
&\quad + 2\|\epsilon_k\|_{M_k}((1+\alpha_k)\|z_k-z\|_{M_k} + \alpha_k\|z_{k-1}-z\|_{M_k}) \\
&\quad + \|\epsilon_k\|^2_{M_k} - \|B\bar{z}_k - Bz\|^2_{2\beta-M_k^{-1}} - \|(\bar{z}_k-\hat{z}_{k+1}) - (B_k\bar{z}_k - B_kz)\|^2_{M_k}\big) \\
&\overset{\text{(v)}}{\le} (1+\eta_k)\big(\|z_k-z\|^2_{M_k} + \alpha_k\|z_k-z_{k-1}\|_{M_k}(\|z_k-z\|_{M_k} + \|z_{k-1}-z\|_{M_k}) \\
&\quad + (1+\alpha_k)\alpha_k\|z_k-z_{k-1}\|^2_{M_k} + 2\|\epsilon_k\|_{M_k}((1+\alpha_k)\|z_k-z\|_{M_k} + \alpha_k\|z_{k-1}-z\|_{M_k}) \\
&\quad + \|\epsilon_k\|^2_{M_k} - \|B\bar{z}_k - Bz\|^2_{2\beta-M_k^{-1}} - \|(\bar{z}_k-\hat{z}_{k+1}) - (B_k\bar{z}_k - B_kz)\|^2_{M_k}\big)\,.
\end{aligned}
\tag{58}
$$

where (i) uses (53), (ii) uses (55), (iii) uses (56) and (57), (iv) uses factorization of the quadratic, and (v) uses the triangle inequality to obtain the bound $\|z_k-z\|_{M_k}-\|z_{k-1}-z\|_{M_k} \le \|z_k-z_{k-1}\|_{M_k}$.

Now, our goal is to conclude boundedness using Lemma 2.7. For simplicity, we set:

$$
\begin{cases}
e_k &:= \alpha_k\|z_k-z_{k-1}\|^2_{M_k} \\
r_k &:= \alpha_k\|z_k-z_{k-1}\|_{M_k} \\
\theta_k &:= \|z_k-z\|_{M_k} \\
m_k &:= \|z_{k-1}-z\|_{M_k} \\
p_k &:= \|B\bar{z}_k - Bz\|^2_{2\beta-M_k^{-1}} \\
q_k &:= \|(\bar{z}_k-\hat{z}_{k+1}) - (B_k\bar{z}_k - B_kz)\|^2_{M_k}\,.
\end{cases}
$$

By Assumption 1, we have $m_k \le (1+\eta_{k-1})\theta_{k-1}$. Without loss of generality, we can assume $0<\eta_k<1$ for any $k\in\mathbb{N}$. Replacing each term in (58) with new corresponding notations, we obtain:

$$
\begin{aligned}
\theta^2_{k+1} &\le (1+\eta_k)\big(\theta_k^2 + r_k(\theta_k + m_k) \\
&\quad + (1+\alpha_k)e_k + 2\|\epsilon_k\|_{M_k}((1+\alpha_k)\theta_k + \alpha_k m_k) + \|\epsilon_k\|^2_{M_k} - p_k - q_k\big) \\
&\le (1+\eta_k)\big(\theta_k^2 + r_k(\theta_k + (1+\eta_{k-1})\theta_{k-1}) \\
&\quad + (1+\alpha_k)e_k + 2\|\epsilon_k\|_{M_k}((1+\alpha_k)\theta_k + \alpha_k(1+\eta_{k-1})\theta_{k-1}) + \|\epsilon_k\|^2_{M_k}\big) \\
&\le (1+\eta_k)\big(\theta_k^2 + r_k(\theta_k + 2\theta_{k-1}) + (1+\Lambda)e_k + 2\|\epsilon_k\|_{M_k}((1+\Lambda)\theta_k + 2\Lambda\theta_{k-1}) + \|\epsilon_k\|^2_{M_k}\big)\,,
\end{aligned}
\tag{59}
$$

where the last inequality uses $0<\alpha_k\le\Lambda$ and $0<\eta_k<1$. Now, we claim that θ_k is bounded in two steps. We introduce an auxiliary bounded sequence $(C_k)_{k\in\mathbb{N}}$ (step 1) such that $\theta_k \le C_k$ for any $k\in\mathbb{N}$ (step 2). The boundedness of θ_k follows from that of C_k.

Step1: We construct a sequence C_k as the following:

$$
\begin{cases}
C_0 &= \max\{\theta_0, 1\}\,, \\
C_k &= (1+\eta_k)C_k + \nu_k\,,
\end{cases}
\tag{60}
$$

where $\nu_k = (1+\eta_k)((1+\Lambda)e_k + 2r_k + (1+3\Lambda)\|\epsilon_k\|_{M_k})$. From our assumptions, it holds that $(M_k)_{k\in\mathbb{N}}$ is bounded from above, $(r_k)_{k\in\mathbb{N}} \in \ell^1_+(\mathbb{N})$, $(e_k)_{k\in\mathbb{N}} \in \ell^1_+(\mathbb{N})$ and $(\|\epsilon_k\|)_{k\in\mathbb{N}} \in \ell^1_+(\mathbb{N})$, which implies $(\nu_k)_{k\in\mathbb{N}} \in \ell^1_+(\mathbb{N})$. Using Lemma 2.7, we obtain the convergence of C_k to some $C_\infty < +\infty$.
Step2: From the update step of (60), we observe that $(C_k)_k$ is a non-decreasing sequence and $C_k \ge 1$ for any $k \in \mathbb{N}$. We claim that for each k, $\theta_k \le C_k$. We argument by induction. Clearly, we have $\theta_0 \le C_0$. Assume $\theta_i \le C_i$ holds true for $i \le k$. Then, (59) yields that

$$\begin{aligned}
\theta_{k+1}^2 &\le (1+\eta_k)\big(C_k^2 + r_k(C_k + 2C_{k-1}) + (1+\Lambda)e_k + 2\|\epsilon_k\|_{M_k}((1+\Lambda)C_k + 2\Lambda C_{k-1}) + \|\epsilon_k\|^2_{M_k}\big)\\
&\overset{(*)}{\le} (1+\eta_k)\big(C_k^2 + 4r_kC_k + 2(1+\Lambda)e_kC_k + 2(1+3\Lambda)\|\epsilon_k\|_{M_k}C_k + \|\epsilon_k\|^2_{M_k}\big)\\
&\le (1+\eta_k)\big(C_k + 2r_k + (1+\Lambda)e_k + (1+3\Lambda)\|\epsilon_k\|_{M_k}\big)^2\,,
\end{aligned} \tag{61}$$

where $(*)$ uses $C_k \ge C_{k-1} \ge 1$ and $r_k > 0$. By the definition of C_{k+1}, we obtain

$$\begin{aligned}
\theta_{k+1} &\overset{\text{(i)}}{\le} \sqrt{1+\eta_k}\big(C_k + 2r_k + (1+\Lambda)e_k + (1+3\Lambda)\|\epsilon_k\|_{M_k}\big)\\
&\overset{\text{(ii)}}{\le} (1+\eta_k)C_k + (1+\eta_k)(2r_k + (1+\Lambda)e_k + (1+3\Lambda)\|\epsilon_k\|_{M_k})\\
&\overset{\text{(iii)}}{\le} (1+\eta_k)C_k + \nu_k\\
&= C_{k+1}\,,
\end{aligned} \tag{62}$$

where (i) uses (61), (ii) holds true since $(1+\eta_k) > 1$ and (iii) uses definition of ν_k. This concludes the induction, and we deduce that θ_k is bounded and therefore, z_k and $\bar{z}_k$ are both bounded.
Weak convergence:
This part of the proof is adapted from the one for [11, Theorem 4.1]. Since θ_k is bounded, we set $\zeta := \sup_{k\in\mathbb{N}} \theta_k$. The last inequality in (58) implies that

$$\begin{aligned}
\theta_{k+1}^2 &\le (1+\eta_k)\big(\theta_k^2 + r_k(\zeta + 2\zeta) + (1+\Lambda)e_k + 2(1+3\Lambda)\|\epsilon_k\|_{M_k}\zeta + \|\epsilon_k\|^2_{M_k} - p_k - q_k\big)\\
&\le (1+\eta_k)\big(\theta_k^2 + 3r_k\zeta + (1+\Lambda)e_k + 2(1+3\Lambda)\|\epsilon_k\|_{M_k}\zeta + \|\epsilon_k\|^2_{M_k} - p_k - q_k\big)\\
&\le \theta_k^2 + \eta_k\theta_k^2 + (1+\eta_k)\big(3r_k\zeta + (1+\Lambda)e_k + (2+6\Lambda)\|\epsilon_k\|_{M_k}\zeta + \|\epsilon_k\|^2_{M_k}\big) - p_k - q_k\\
&\le \theta_k^2 + \underbrace{\eta_k\zeta^2 + 2(3r_k\zeta + (1+\Lambda)e_k + (2+6\Lambda)\|\epsilon_k\|_{M_k}\zeta + \|\epsilon_k\|^2_{M_k})}_{\delta_k} - p_k - q_k\,.
\end{aligned} \tag{63}$$

We set $\delta_k := \eta_k\zeta^2 + 2(3r_k\zeta + (1+\Lambda)e_k + (2+6\Lambda)\|\epsilon_k\|_{M_k}\zeta + \|\epsilon_k\|^2_{M_k})$ and observe that $(\delta_k)_{k\in\mathbb{N}} \in \ell^1_+(\mathbb{N})$. Now, (63) yields that

$$\theta_{k+1}^2 \le \theta_k^2 + \delta_k\,. \tag{64}$$

Using (64) and Lemma 2.7, we obtain the convergence of $\theta_k^2 = \|z_k - z\|^2_{M_k}$ for any $z \in zer(A+B)$. Rearranging (63) to $p_k \le \theta_k^2 - \theta_{k+1}^2 + \delta_k$, using Assumption 1 and summing it for $k = 0, \cdots, N$, we obtain

$$\epsilon \sum_{k=0}^{N} \|B\bar{z}_k - Bz\|^2 \le \sum_{k=0}^{N} \|B\bar{z}_k - Bz\|^2_{2\beta - M_k^{-1}} = \sum_{k=0}^{N} p_k \le \theta_0^2 - \theta_N^2 + \sum_{k=0}^{N}\delta_k \le \zeta^2 + \sum_{k=0}^{N}\delta_k\,. \tag{65}$$

Since $(\delta_k)_{k\in\mathbb{N}} \in \ell^1_+(\mathbb{N})$, by taking limit as $N \to +\infty$, we obtain

$$\sum_{k\in\mathbb{N}} \|B\bar{z}_k - Bz\|^2 \le \frac{1}{\epsilon^2}(\zeta^2 + \sum_{k\in\mathbb{N}}\delta_k) < +\infty\,. \tag{66}$$

Similarly, we obtain from (63) using $q_k \le \theta_k^2 - \theta_{k+1}^2 + \delta_k$ that

$$\sum_{k\in\mathbb{N}} \|(\bar{z}_k - \hat{z}_{k+1}) - (B_k\bar{z}_k - B_kz)\|^2_{M_k} < +\infty\,. \tag{67}$$

Set z^* as an arbitrary weak sequential cluster point of $(z_k)_{k\in\mathbb{N}}$, namely, a subsequence $z_{k_n} \rightharpoonup z^*$ as $n \to \infty$.

In order to obtain weak convergence of z_k, by Proposition 2.5 with $\varphi(t) = t^2$ and (64) and Assumption 1, it suffices to show that $z^* \in zer(A+B)$. It follows from the selection of α_k that:

$$\|\bar{z}_k - z_k\| \le \alpha_k \|z_k - z_{k-1}\| \to 0 \,. \tag{68}$$

Thus, (68) yields $\bar{z}_{k_n} \rightharpoonup z^*$. From (66), we obtain that $B\bar{z}_{k_n} \to Bz$ as $n \to \infty$. Since B is cocoercive, it is maximally monotone and we can use the weak strong graph closedness of B in Proposition 2.2 to infer that $(z^*, Bz) \in \operatorname{Graph} B$, i.e. $Bz \in Bz^*$. However, since B is single valued, we obtain $Bz^* = Bz$ and hence $B\bar{z}_{k_n} \to Bz^*$. Setting $u_k := M_k(\bar{z}_k - \hat{z}_{k+1}) - B\bar{z}_k$, by definition of the resolvent $J_A^{M_k}$, we have $u_k \in A(\hat{z}_{k+1})$ for all $k \in \mathbb{N}$. From (67), we obtain as $k \to +\infty$,

$$\begin{aligned} \|u_k + Bz^*\| &= \|M_k(\bar{z}_k - \hat{z}_{k+1} - B_k\bar{z}_k + B_k z^*)\| \\ &\le C\|\bar{z}_k - \hat{z}_{k+1} - B_k\bar{z}_k + B_k z^*\| \\ &\le \frac{C}{\sigma}\|\bar{z}_k - \hat{z}_{k+1} - B_k\bar{z}_k + B_k z^*\|_{M_k} \to 0 \,. \end{aligned} \tag{69}$$

Furthermore, from (66) and (67), we have

$$\begin{aligned} \|\bar{z}_k - \hat{z}_{k+1}\| &\le \|\bar{z}_k - \hat{z}_{k+1} - B_k\bar{z}_k + B_k z^*\| + \|B_k\bar{z}_k - B_k z^*\| \\ &\le \|\bar{z}_k - \hat{z}_{k+1} - B_k\bar{z}_k + B_k z^*\| + \frac{1}{\sigma}\|B\bar{z}_k - Bz^*\| \to 0 \,. \end{aligned} \tag{70}$$

Therefore, together with (70), $\bar{z}_{k_n} \rightharpoonup z^*$ implies $\hat{z}_{k_n+1} \rightharpoonup z^*$ as $n \to \infty$. Now we already have $u_{k_n} \to -Bz^*$ as $n \to \infty$ and

$$(\forall k \in \mathbb{N}): \quad (\hat{z}_{k_n+1}, u_{k_n}) \in \operatorname{Graph} A \,. \tag{71}$$

Since A is maximally monotone and using Proposition 2.2, we infer that $-Bz^* \in Az^*$, hence $z^* \in zer(A+B)$. As mentioned above, the result follows from Proposition 2.5 with $\varphi(t) = t^2$.

Convergence rate:

In the following part, we are going to show the convergence rate of Algorithm 1: Assume $\epsilon_k \equiv 0$ for $k \in \mathbb{N}$ and either $\gamma_A > 0$ or $\gamma_B > 0$. Because of (52), Assumption 1 and Lipschitz continuity of $J_A^{M_k}$, we obtain for any $z \in zer(A+B)$ that

$$\begin{aligned} \|z_{k+1} - z\|^2_{M_{k+1}} &\le (1+\eta_k)\|z_{k+1} - z\|^2_{M_k} \\ &\le (1+\eta_k)\Big(\frac{1}{1+\frac{\gamma_A}{C}}\Big)^2 \|(\bar{z}_k - B_k\bar{z}_k) - (z - B_k z)\|^2_{M_k} \\ &\overset{\text{(i)}}{\le} (1+\eta_k)\Big(\frac{1}{1+\frac{\gamma_A}{C}}\Big)^2 \big(\|\bar{z}_k - z\|^2_{M_k} - \|B\bar{z}_k - Bz\|^2_{\beta - M_k^{-1}} - \gamma_B\|\bar{z}_k - z\|^2\big) \\ &\overset{\text{(ii)}}{\le} (1+\eta_k)\Big(\frac{1}{1+\frac{\gamma_A}{C}}\Big)^2 (1 - \frac{\gamma_B}{C})\big(\|\bar{z}_k - z\|^2_{M_k}\big) \\ &\overset{\text{(iii)}}{=} (1+\eta_k)\frac{(1-\frac{\gamma_B}{C})}{(1+\frac{\gamma_A}{C})^2}\big((1+\alpha_k)\|z_k - z\|^2_{M_k} - \alpha_k\|z_{k-1} - z\|^2_{M_k} \\ &\quad + (1+\alpha_k)\alpha_k\|z_k - z_{k-1}\|^2_{M_k}\big) \\ &= (1+\eta_k)\frac{(1-\frac{\gamma_B}{C})}{(1+\frac{\gamma_A}{C})^2}\big(\|z_k - z\|^2_{M_k} + \alpha_k(\|z_k - z\|^2_{M_k} - \|z_{k-1} - z\|^2_{M_k}) \\ &\quad + (1+\alpha_k)\alpha_k\|z_k - z_{k-1}\|^2_{M_k}\big) \\ &\overset{\text{(iv)}}{=} (1+\eta_k)\frac{(1-\frac{\gamma_B}{C})}{(1+\frac{\gamma_A}{C})^2}\big(\|z_k - z\|^2_{M_k} + \alpha_k\|z_k - z_{k-1}\|_{M_k}(\|z_k - z\|_{M_k} \\ &\quad + \|z_{k-1} - z\|_{M_k}) + (1+\alpha_k)\alpha_k\|z_k - z_{k-1}\|^2_{M_k}\big) \\ &\overset{\text{(v)}}{=} (1+\eta_k)\frac{(1-\frac{\gamma_B}{C})}{(1+\frac{\gamma_A}{C})^2}\big(\|z_k - z\|^2_{M_k} + \alpha_k\|z_k - z_{k-1}\|_{M_k}(\|z_k - z\|_{M_k} \\ &\quad + (1+\eta_{k-1})\|z_{k-1} - z\|_{M_{k-1}}) + (1+\alpha_k)\alpha_k\|z_k - z_{k-1}\|^2_{M_k}\big) \\ &\overset{\text{(vi)}}{=} (1+\eta_k)\frac{(1-\frac{\gamma_B}{C})}{(1+\frac{\gamma_A}{C})^2}\big(\|z_k - z\|^2_{M_k} + 3\alpha_k\zeta\|z_k - z_{k-1}\|_{M_k} \\ &\quad + (1+\Lambda)\alpha_k\|z_k - z_{k-1}\|^2_{M_k}\big) \end{aligned} \tag{72}$$

where (i) uses (II), (ii) uses the fact M_k is bounded uniformly and the assumption that $M_k - \frac{1}{\beta}I \in \mathcal{S}_\epsilon(\mathcal{H})$, (iii) uses (56), (iv) uses factorization of the quadratic and uses the triangle inequality to obtain the bound $\|z_k - z\|_{M_k} - \|z_{k-1} - z\|_{M_k} \leq \|z_k - z_{k-1}\|_{M_k}$, (v) uses Assumption 1 and (vi) uses boundedness of α_k and $\|z_k - z\|_{M_k}$. Since either $\gamma_A > 0$ or $\gamma_B > 0$, $\frac{1-\frac{\gamma_B}{C}}{(1+\frac{\gamma_A}{C})^2} < 1$. Then there exists sufficient large $K_0 > 0$ such that for any $k > K_0$, $(1 + \eta_k)\left(\frac{1-\frac{\gamma_B}{C}}{(1+\frac{\gamma_A}{C})^2}\right) < 1 - \xi < 1$ for some $\xi \in (0, 1)$. Thus, we infer that for any $k > K_0$:

$$\|z_k - z\|^2_{M_k} \leq (1-\xi)^{k-K_0}\|z_{K_0} - z\|^2_{M_{K_0}} + \sum_{i=K_0}^{k-1}(1-\xi)^{k-i}\alpha_i(3\zeta\|z_i - z_{i-1}\|_{M_i} + (1+\Lambda)\|z_i - z_{i-1}\|^2_{M_i})\,. \tag{73}$$

Let $\Theta = 3\zeta + (1 + \Lambda)$. Therefore (73) can be simplified as the following:

$$\|z_k - z\|^2_{M_k} \leq (1-\xi)^{k-K_0}\|z_{K_0} - z\|^2_{M_{K_0}} + \sum_{i=K_0}^{k-1}\Theta(1-\xi)^{k-i}\alpha_i \max\{\|z_i - z_{i-1}\|_{M_i}, \|z_i - z_{i-1}\|^2_{M_i}\}\,. \tag{74}$$

Since z_k is bounded and $M_k \in S_\sigma(\mathcal{H})$, it follows that

$$\|z_k - z\|^2 \leq \tfrac{1}{\sigma}(1-\xi)^{k-K_0}\|z_{K_0} - z\|^2_{M_{K_0}} + O\Big(\sum_{i=K_0}^{k-1}(1-\xi)^{k-i}\alpha_i\Big)\,. \tag{75}$$

Furthermore, if $\alpha_i \equiv 0$, for $k > K_0$ and for any $z \in zer(A + B)$, we obtain linear convergence:

$$\|z_{k+1} - z\|^2 \leq \frac{1}{\sigma}(1-\xi)^{k-K_0}\|z_{K_0} - z\|^2_{M_{K_0}}\,. \tag{76}$$

If $\alpha_k \neq 0$, $\alpha_k = O(\frac{1}{k^2})$ and K_0 large enough, then $\|z_{k+1} - z\|^2$ converges in the rate of $O(\frac{1}{k})$ for $k > K_0$ according to [29, Lemma 2.2.4 (Chung)]; if $\alpha_k \neq 0$ and $\alpha_k = O(q^k)$ for $q = 1 - \xi$ and $k > K_0$, then $\|z_{k+1} - z\|^2$ converges in the rate of $O(kq^k)$ for $k > K_0$ since (75).

□

A.4 Proof of Theorem 3.2

Proof. For simplicity, we set

$$\begin{cases} \hat{z}_k := J_A^{M_k}(z_k - M_k^{-1}Bz_k), \text{i.e.}, \tilde{z}_k = \hat{z} + \epsilon_k\,, \\ \delta_k := \max\{1, \rho\}\|\epsilon_k\|\,, \end{cases} \tag{77}$$

where $\rho = \sqrt{\frac{C}{\sigma}\frac{(C+\frac{1}{\beta})}{2c}}$ and $(\delta_k)_{k\in\mathbb{N}} \in \ell^1_+(\mathbb{N})$. We set z^* such that $-Bz^* \in Az^*$.

Boundedness:

We claim that by choosing proper t_k for each $k \in \mathbb{N}$, we have

$$\|z_{k+1} - z^*\| \leq \|z_k - z^*\| + \delta_k. \tag{78}$$

Note that if (78) is satisfied, it follows from Lemma 2.7 that $\|z_k - z^*\|$ is bounded and converges.

To stress the relation between z_{k+1} and t_k, we define $z(t_k) := z_k - t_k[(M_k - B)(z_k - \tilde{z}_k)]$ and we will use z_{k+1} and $z(t_k)$ interchangeably. We also set $\gamma_k(t_k) := (\delta_k + ||z_k - z^*||)^2 - ||z(t_k) - z^*||^2$.

In order to prove (78), it is sufficient to show that for proper t_k at each iterate, $\gamma_k(t_k) > 0$. It results from the definition of $\gamma_k(t_k)$ that

$$\begin{aligned}\gamma_k(t_k) &= (||z_k - z^*|| + \delta_k)^2 - ||z(t_k) - z^*||^2 \\ &= \langle z_k - z^* + z(t_k) - z^*, z_k - z(t_k)\rangle + 2\delta_k||z_k - z^*|| + \delta_k^2 \\ &= \underbrace{2t_k\,\langle z_k - z^*, (M_k - B)(z_k - \tilde{z}_k)\rangle}_{\text{(I)}} - \underbrace{t_k^2||(M_k - B)(z_k - \tilde{z}_k)||^2}_{\text{(II)}} \\ &\quad + 2\delta_k||z_k - z^*|| + \delta_k^2\,.\end{aligned} \tag{79}$$

We need several useful properties to estimate (I) and (II).

It follows from the definition of $\hat{z}_k$ that $-M_k(\hat{z}_k - z_k) - Bz_k \in A\hat{z}_k$, and γ_A-strong monotonicity of A implies that

$$
\begin{aligned}
\gamma_A\|\hat{z}_k - z^*\|^2 &\le \langle \hat{z}_k - z^*, -M_k(\hat{z}_k - z_k) - Bz_k + Bz^*\rangle \\
&= \langle \hat{z}_k - z^*, M_k(z_k - \hat{z}_k) - Bz_k + B\hat{z}_k - B\hat{z}_k + Bz^*\rangle \\
&= \langle \hat{z}_k - z^*, M_k(z_k - \hat{z}_k) - Bz_k + B\hat{z}_k\rangle - \langle \hat{z}_k - z^*, B\hat{z}_k - Bz^*\rangle \,.
\end{aligned} \tag{80}
$$

We deduce by (80) and γ_B-strong monotonicity of B that

$$
\begin{aligned}
\langle \hat{z}_k - z^*, M_k(z_k - \hat{z}_k) - Bz_k + B\hat{z}_k\rangle &\ge \langle \hat{z}_k - z^*, B\hat{z}_k - Bz^*\rangle + \gamma_A\|\hat{z}_k - z^*\|^2 \\
&\ge \gamma_B\|\hat{z}_k - z^*\|^2 + \gamma_A\|\hat{z}_k - z^*\|^2 \,.
\end{aligned} \tag{81}
$$

$J_A^{M_k}$ is Lipschitz since A is monotone. Therefore,

$$
\begin{aligned}
\|\hat{z}_k - z^*\|_{M_k}^2 &\le \|(z_k - B_k z_k) - (z^* - B_k z^*)\|_{M_k}^2 \\
&= \|z_k - z^*\|_{M_k}^2 - 2\,\langle z_k - z^*, B_k z_k - B_k z^*\rangle_{M_k} + \|B_k z_k - B_k z^*\|_{M_k}^2 \\
&\le \|z_k - z^*\|_{M_k}^2 - \|Bz_k - Bz\|_{2\beta - M_k^{-1}} \\
&\overset{(*)}{\le} \|z_k - z^*\|_{M_k}^2 \,,
\end{aligned} \tag{82}
$$

where $(*)$ uses Assumption 2 that $M_k - \frac{1}{\beta}I \in \mathcal{S}_c(\mathcal{H})$. Using the assumption that $M_k - \frac{1}{\beta}I \in \mathcal{S}_c(\mathcal{H})$ again, we obtain

$$
\langle z_k - \tilde{z}_k, (M_k - B)(z_k - \tilde{z}_k)\rangle \ge \|z_k - \tilde{z}_k\|^2_{M_k - \frac{1}{\beta}} > 0\,. \tag{83}
$$

Combining (81), (82), the first term (I) in (79) can be estimated by the following:

$$
\begin{aligned}
\text{(I)} &= 2t_k\,\langle z_k - z^*, (M_k - B)(z_k - \tilde{z}_k)\rangle \\
&= 2t_k\,\langle z_k - \tilde{z}_k, (M_k - B)(z_k - \tilde{z}_k)\rangle + 2t_k\,\langle \tilde{z}_k - \hat{z}_k, (M_k - B)(z_k - \tilde{z}_k)\rangle \\
&\quad + 2t_k\,\langle \hat{z}_k - z^*, (M_k - B)(z_k - \hat{z}_k)\rangle + 2t_k\,\langle \hat{z}_k - z^*, (M_k - B)(\hat{z}_k - \tilde{z}_k)\rangle \\
&\overset{\text{(i)}}{\ge} 2t_k\,\langle z_k - \tilde{z}_k, (M_k - B)(z_k - \tilde{z}_k)\rangle + 2t_k\,\langle \tilde{z}_k - \hat{z}_k, (M_k - B)(z_k - \tilde{z}_k)\rangle \\
&\quad + 2t_k(\gamma_A + \gamma_B)\|\hat{z}_k - z^*\|^2 + 2t_k\,\langle \hat{z}_k - z^*, (M_k - B)(\hat{z}_k - \tilde{z}_k)\rangle \\
&\overset{\text{(ii)}}{\ge} 2t_k\,\langle z_k - \tilde{z}_k, (M_k - B)(z_k - \tilde{z}_k)\rangle - 2t_k\|\epsilon_k\|\,\|(M_k - B)(z_k - \tilde{z}_k)\| \\
&\quad + 2t_k(\gamma_A + \gamma_B)\|\hat{z}_k - z^*\|^2 - 2t_k\|(M_k - B)\epsilon_k\|_{M_k^{-1}}\|\hat{z}_k - z^*\|_{M_k} \\
&\overset{\text{(iii)}}{\ge} 2t_k\,\langle z_k - \tilde{z}_k, (M_k - B)(z_k - \tilde{z}_k)\rangle - \|\epsilon_k\|^2 - t_k^2\|(M_k - B)(z_k - \tilde{z}_k)\|^2 \\
&\quad + 2t_k(\gamma_A + \gamma_B)\|\hat{z}_k - z^*\|^2 - 2\tfrac{1}{\sqrt{\sigma}}(C + \tfrac{1}{\beta})t_k\|\epsilon_k\|\,\|z_k - z^*\|_{M_k} \\
&\overset{\text{(iv)}}{\ge} 2t_k\,\langle z_k - \tilde{z}_k, (M_k - B)(z_k - \tilde{z}_k)\rangle - \|\epsilon_k\|^2 - t_k^2\|(M_k - B)(z_k - \tilde{z}_k)\|^2 \\
&\quad + 2t_k(\gamma_A + \gamma_B)\|\hat{z}_k - z^*\|^2 - 2\sqrt{\tfrac{C}{\sigma}}(C + \tfrac{1}{\beta})t_k\|\epsilon_k\|\,\|z_k - z^*\|\,,
\end{aligned} \tag{84}
$$

where (i) uses (81), (ii) uses Cauchy inequality and (iii) uses (82), $2ab \le a^2 + b^2$ and Assumption 2. We set $b_k := \langle z_k - \tilde{z}_k, (M_k - B)(z_k - \tilde{z}_k)\rangle$ and $a_k := \|(M_k - B)(z_k - \tilde{z}_k)\|^2$. The definition of δ_k yields that $\delta_k \ge \|\epsilon_k\|^2$ and it follows from (84) that:

$$
\begin{aligned}
\gamma_k(t_k) &\ge 2t_k b_k - 2t_k^2 a_k + 2\delta_k\|z_k - z^*\| + \delta_k^2 - \|\epsilon_k\|^2 \\
&\quad - 2\sqrt{\tfrac{C}{\sigma}}(C + \tfrac{1}{\beta})t_k\|\epsilon_k\|\,\|z_k - z^*\| + 2t_k(\gamma_A + \gamma_B)\|\hat{z}_k - z^*\|^2 \\
&\ge \underbrace{2t_k b_k - 2t_k^2 a_k}_{\text{(III)}} + 2t_k(\gamma_A + \gamma_B)\|\hat{z}_k - z^*\|^2 + \underbrace{2(\delta_k - \sqrt{\tfrac{C}{\sigma}}(C + \tfrac{1}{\beta})t_k)\|\epsilon_k\|\,\|z_k - z^*\|}_{\text{(IV)}}\,.
\end{aligned} \tag{85}
$$

We continue to find a proper t_k such that $\gamma_k(t_k) > 0$. This goal boils down to ensuring both (III) and (IV) are positive.

Let $t_k = \frac{b_k}{2a_k}$. We first show $t_k = \frac{b_k}{2a_k}$ is the proper value to make sure (III) is positive. From (83), we observe that $b_k > 0$. Since $a_k > 0$ and $b_k > 0$, the quadratic term (III) in (85) will be zero for $t_k = 0$ or $t_k = \frac{b_k}{a_k}$ and will be strictly positive for any $t_k \in (0, b_k/a_k)$ with the maximum value obtained at $t_k = \frac{b_k}{2a_k}$. As a result, (III) is strictly positive.

Second, we will show (IV) is positive when $t_k = \frac{b_k}{2a_k}$. We observe that $0 < t_k = \frac{b_k}{2a_k} < \frac{1}{2c}$ for Assumption 2. Thus, the definition of δ_k and $t_k = \frac{b_k}{2a_k}$ imply that

$$\begin{aligned}
(\text{IV}) &= 2(\delta_k - \sqrt{\tfrac{C}{\sigma}}(C + \tfrac{1}{\beta})t_k)\|\epsilon_k\|\|z_k - z^*\| \\
&\geq (2\delta_k - \sqrt{\tfrac{C}{\sigma}}\tfrac{(C+\frac{1}{\beta})}{c})\|\epsilon_k\|\|z_k - z^*\| \\
&\geq 0\,.
\end{aligned} \tag{86}$$

Since (III) and (IV) both are positive when $t_k = \frac{b_k}{2a_k}$, (85) yields:

$$\begin{aligned}
\gamma_k(t_k) &\geq 2t_k b_k - 2t_k^2 a + 2t_k(\gamma_A + \gamma_B)\|\hat{z}_k - z^*\|^2 + (2\delta_k - \sqrt{\tfrac{C}{\sigma}}\tfrac{(C+\frac{1}{\beta})}{c})\|\epsilon_k\|\|z_k - z^*\| \\
&\geq \frac{b_k^2}{2a_k} + 2t_k(\gamma_A + \gamma_B)\|\hat{z}_k - z^*\|^2 \\
&> 0\,,
\end{aligned} \tag{87}$$

It results from (87) and the definition of $\gamma_k(t_k)$ that for each $k \in \mathbb{N}$, $(||z_k - z^*|| + \delta_k) \geq ||z(t_k) - z^*|| = ||z_{k+1} - z^*||$. We conclude that if $t_k = \frac{b_k}{2a_k}$, then $\gamma_k(t_k) > 0$ for all $k \in \mathbb{N}$ and the sequence $\|z_k - z^*\|$ is bounded and converges as $k \to +\infty$ by using Lemma 2.7.

Weak convergence:

The sequence $(z_k)_{k\in\mathbb{N}}$ generated by Algorithm 2 is bounded and $\|z_k - z\|$ converges as $k \to \infty$ and $\gamma_k(t_k)$ converges to zero as $k \to \infty$ for all $z \in zer(A + B)$. Set z^* as an arbitrary weak sequential cluster point of $(z_k)_{k\in\mathbb{N}}$ and there exists a subsequence $(z_{k_n})_{n\in\mathbb{N}}$ such that $z_{k_n} \rightharpoonup z^*$.

In order to obtain weak convergence of z_k, by Proposition 2.5 with $\varphi(t) = t$ and fixed metric $M_k = I$ and (78), it suffices to show that $z^* \in zer(A + B)$.

Using Assumption 2, $(M_k)_{k\in\mathbb{N}}$ is bounded uniformly by C. Together with boundedness of operator B, we obtain a_k is bounded by $(C + \frac{1}{\beta})^2\|z_k - \tilde{z}_k\|^2$ for each $k \in \mathbb{N}$. Using Assumption 2, we obtain that $b_k \geq \|z_k - \tilde{z}_k\|^2_{M_k - \frac{1}{\beta}I} \geq c\|z_k - \tilde{z}_k\|^2$. By definition of t_k and (87), we have $\gamma_k(t_k) \geq \frac{c}{2(C+\frac{1}{\beta})^2}\|z_k - \tilde{z}_k\|^2$. Since $\gamma_k(t_k) \to 0$, $\|z_k - \tilde{z}_k\| \to 0$ as $k \to \infty$. Moreover, since $\epsilon_k \to 0$, $\|z_k - \hat{z}_k\| \to 0$. We set $u_k := M_k(z_k - \hat{z}_k) + B\hat{z}_k - Bz_k$. Therefore, we obtain that $u_k \to 0$ as $k \to +\infty$ and $\hat{z}_{k_n} \rightharpoonup z^*$ as $n \to +\infty$. We observe that $u_k \in A\hat{z}_k + B\hat{z}_k$. Then, by using Proposition 2.2 and the fact that $A+B$ is maximally monotone, we conclude that $0 \in Az^* + Bz^*$ and $\|z_k - z^*\|$ decreases since $z^* \in zer(A+B)$. As mentioned above, the result follows from Proposition 2.5 with $\varphi(t) = t$.

Linear convergence rate:

If we assume $\epsilon_k \equiv 0$, then $\hat{z}_k = \tilde{z}_k$ and $\delta_k \equiv 0$. Therefore, from (87) we can obtain an estimation for $\gamma_k(t_k)$ when $t_k = \frac{b_k}{2a_k}$:

$$\gamma_k(t_k) = \|z_k - z^*\|^2 - \|z_{k+1} - z^*\|^2 \geq \frac{b_k^2}{2a_k} + 2t_k(\gamma_A + \gamma_B)\|\hat{z}_k - z^*\|^2\,. \tag{88}$$

The following part is to derive linear convergence for the case that either $\gamma_A > 0$ or $\gamma_B > 0$. The definition of b_k and that of a_k yield the following estimation for t_k:

$$t_k = \frac{b_k}{2a_k} = \frac{\langle z_k - \hat{z}_k, (M_k - B)(z_k - \hat{z}_k)\rangle}{2\|(M_k - B)(z_k - \hat{z}_k)\|^2} \overset{(i)}{\geq} \frac{\|z_k - \hat{z}_k\|^2_{M_k - \frac{1}{\beta}}}{2(C + \frac{1}{\beta})^2\|z_k - \hat{z}_k\|^2} \overset{(ii)}{>} \frac{c}{2(C + \frac{1}{\beta})^2}, \tag{89}$$

where both (i) and (ii) use Assumption 2. For convenience, we denote $\frac{c}{2(C+\frac{1}{\beta})^2}$ by δ. Using Assumption 2 again, we have the estimation for the first term at the right hand side of (88):

$$\tfrac{b_k^2}{2a_k} \geq \delta\|z_k - \hat{z}_k\|^2_{M_k - \frac{1}{\beta}I} > c\delta\|z_k - \hat{z}_k\|^2\,. \tag{90}$$

Furthermore, combining (88) with (89), (90) and the definition $\gamma_k(t_k) := \|z_k - z^*\|^2 - \|z_{k+1} - z^*\|^2$, we obtain

$$\begin{aligned}\|z_{k+1} - z^*\|^2 &\leq \|z_k - z^*\|^2 - 2(\gamma_A + \gamma_B)\delta\|\hat{z}_k - z^*\|^2 - c\delta\|z_k - \hat{z}_k\|^2 \\ &\leq \|z_k - z^*\|^2 - \tfrac{1}{2}\min\{2(\gamma_A + \gamma_B)\delta, c\delta\}(2\|\hat{z}_k - z^*\|^2 + 2\|z_k - \hat{z}_k\|^2) \\ &\overset{(i)}{\leq} \|z_k - z^*\|^2 - \tfrac{1}{2}\min\{2(\gamma_A + \gamma_B)\delta, c\delta\}(\|\hat{z}_k - z^*\| + \|z_k - \hat{z}_k\|)^2 \\ &\overset{(ii)}{\leq} \|z_k - z^*\|^2 - \tfrac{1}{2}\min\{2(\gamma_A + \gamma_B)\delta, c\delta\}\|z_k - z^*\|^2 \\ &\leq (1 - \tfrac{1}{2}\min\{2(\gamma_A + \gamma_B)\delta, c\delta\})\|z_k - z^*\|^2 ,\end{aligned} \tag{91}$$

where (i) uses inequality $2x^2 + 2y^2 \geq (x+y)^2$ and (ii) uses triangle inequality. Consequently, we obtain linear convergence if $(\gamma_A + \gamma_B) > 0$:

$$\|z_k - z^*\|^2 \leq (1-\xi)^k\|z_0 - z^*\|^2 , \tag{92}$$

where $\xi = \frac{1}{2}\min\{2(\gamma_A + \gamma_B)\delta, c\delta\} > 0$. □

A.5 Proof of Theorem 4.1

Proof. Computing the resolvent operator shows the following equivalences

$$\begin{aligned} & x^* = J_T^V(z) = (I + V^{-1}T)^{-1}(z) \\ \iff & Vz \in (V+T)(x^*) \\ \iff & Mz \in (M+T)(x^*) \pm Q(x^* - z) \\ [y^* = M^{1/2}x^*] \iff & Mz \in (M+T)(M^{-1/2}y^*) \pm Q(M^{-1/2}y^* - z) \\ \iff & Mz \in (M+T)(M^{-1/2}y^*) \pm QM^{-1/2}(y^* - M^{1/2}z) \\ \iff & Mz \in (M^{1/2} + TM^{-1/2})(y^*) \pm QM^{-1/2}(y^* - M^{1/2}z) \\ [W = M^{-1/2}QM^{-1/2}] \iff & M^{1/2}z \in (I + M^{-1/2}TM^{-1/2})(y^*) \pm W(y^* - M^{1/2}z) . \end{aligned} \tag{93}$$

Since existence of x^* is guaranteed by the properties of J_T^V, Lemma 2.6 yields the existence of a unique primal-dual pair (x^*, u^*) that satisfies the equivalent relations in Lemma 2.6 with $B := \pm W \circ \tau_{M^{1/2}z}$ and $A := \tau_{M^{1/2}z} \circ (I + M^{-1/2}TM^{-1/2})$. The mapping B is single-valued and , as T is a maximally monotone operator and M is positive definite, $A^{-1} = J_{M^{-1/2}TM^{-1/2}} \circ \tau_{-M^{1/2}z}$ is single valued. Therefore, the solution of J_T^V can be computed by finding u^* such that

$$0 \in B^{-1}u^* - A^{-1}(-u^*) = [(\pm W)^{-1} + M^{1/2}z] - J_{M^{-1/2}TM^{-1/2}}(M^{1/2}z - u^*) \tag{94}$$

and the using

$$x^* = M^{-1/2}y^* \quad \text{and} \quad y^* = A^{-1}(-u^*) = J_{M^{-1/2}TM^{=1/2}}(M^{1/2}z - u^*) . \tag{95}$$

Substituting $u^* = M^{-1/2}v^*$ in both problems, multiplying the former one from left with $M^{-1/2}$, and using $M^{-1/2}W^{-1}M^{-1/2} = Q^+$ leads to

$$\begin{cases} 0 \in \pm Q^+v^* + z - M^{-1/2} \circ J_{M^{-1/2}TM^{-1/2}} \circ M^{1/2}(z - M^{-1}v^*) \\ x^* = M^{-1/2} \circ J_{M^{-1/2}TM^{-1/2}} \circ M^{1/2}(z - M^{-1}v^*) . \end{cases} \tag{96}$$

Since a solution to J_T^V exists, there exists $v^* \in \operatorname{im}(Q)$ that satisfies the inclusion. Given the isomorphism $U: \mathbb{R}^r \to \operatorname{im}(Q)$, which can be realized using r linearly independent $u_1, \cdots, u_r \in \mathcal{H}$ by $\alpha \to \sum_{i=1}^r \alpha_i u_i$, the inclusion problem is equivalent to finding the unique root $\alpha^* \in \mathbb{R}^r$ of $l(\alpha)$, where U^* denotes the adjoint of U and for the case "$-$" of "$\pm$" we substituted α by $-\alpha$.

The following shows that $l(\alpha)$ is Lipschitz continuous with constant $\|U^*Q^+U\| + \|M^{-1/2}U\|^2$:(We abbreviate $J_{M^{-1/2}TM^{-1/2}}$ by J in the following)

$$\begin{aligned} &\langle l(\alpha) - l(\beta), \alpha - \beta\rangle \\ =&\|\alpha - \beta\|^2_{U^*Q^+U} - \left\langle J(M^{1/2}z \mp M^{-1/2}U\alpha) - J(M^{1/2}z \mp M^{-1/2}U\beta), M^{-1/2}U(\alpha - \beta)\right\rangle \\ \leq&\|U^*Q^+U\|\|\alpha - \beta\|^2 + \|M^{-1/2}U\|^2\|\alpha - \beta\|^2 , \end{aligned} \tag{97}$$

where, in the last line, we use the 1-Lipschitz continuity (non-expansive) of J.

The following shows strict monotonicity of l. We rewrite $l(\alpha)$ as follows:

$$\begin{aligned} l(\alpha) &= U^*Q^+U\alpha + U^*M^{-1/2}(M^{1/2}z - J_{M^{-1/2}TM^{-1/2}}(M^{1/2}z \mp M^{-1/2}U\alpha)) \\ &= U^*Q^+U\alpha \pm U^*M^{-1}U\alpha + U^*M^{-1/2}(I - J_{M^{-1/2}TM^{-1/2}})(M^{1/2}z \mp M^{-1/2}U\alpha) \\ &= U^*(Q^+ \pm M^{-1})U\alpha + U^*M^{-1/2}J_{M^{-1/2}T^{-1}M^{-1/2}}(M^{1/2}z \mp M^{-1/2}U\alpha)\,. \end{aligned} \tag{98}$$

Using the 1-cocoercivity of $J_{M^{1/2}T^{-1}M^{1/2}}$, the function $l(\alpha)$ can be seen to be strictly monotone if $\alpha \mapsto U^*(Q^+ \pm M^{-1})U\alpha$ is strictly monotone. This fact is clear for the case "+" of "±". Therefore, in the remainder, we show strictly monotonicity of $\alpha \mapsto U^*(Q^+ - M^{-1})U\alpha = U^*M^{-1/2}(M^{1/2}Q^+M^{1/2} - I)M^{-1/2}U\alpha$. We observe $M - Q \in \mathcal{S}_0(\mathcal{H})$ implies that $\|M^{-1/2}QM^{-1/2}\| < 1$ and by $1 \le \|AA^{-1}\| \le \|A\|\|A^{-1}\|$, we conclude that $\|M^{1/2}Q^+M^{1/2}\|_{\mathrm{im}(M^{-1/2}Q)}$ for the restriction of the operator norm to $\mathrm{im}(M^{-1/2}Q)$, hence, $Q^+ - M^{-1} \in \mathcal{S}_{++}(\mathcal{H})$.

According to Lemma 2.3, we can replace $M^{-1/2} \circ J_{M^{-1/2}TM^{-1/2}} \circ M^{1/2}$ with J_T^M. Then we obtain the formula in the statement of Theorem 4.1. □

A.6 Proof of Theorem 4.1.1

Proof. Since $Q = UU^*$ for some $U \in \mathcal{B}(\mathbb{R}^r, \mathcal{H})$, we have

$$\mathrm{im}(Q) = \{UU^*v | v \in \mathcal{H}\} = \{U\alpha | \alpha \in \mathbb{R}^r\}\,. \tag{99}$$

Since Q^+ is pseudo-inverse of Q on $im(Q)$, the following holds for arbitrary $v \in \mathcal{H}$:

$$QQ^+Qv = Qv \iff UU^*Q^+UU^*v = UU^*v \iff UU^*Q^+U\alpha = U\alpha\,. \tag{100}$$

Since the column vectors $\{u_i\}_{i=1,\cdots,r}$ of U are independent with each other, $UU^*Q^+U\alpha = U\alpha$ yields that $U^*Q^+U\alpha = \alpha$. Therefore, the root finding problem in Theorem 4.1 simplifies to (22). □

A.7 Proof of Proposition 4.2

Proof. Our proof relies on the convergence result [15, Theorem 7.5.5]. By the same argument as the one in Appendix B.5 paper [7], we obtain $\partial^c l(\alpha^*)$ is non-singular. If $l(\alpha)$ is tame, then by [8, Theorem 1], $l(\alpha)$ is semi-smooth. In order to apply this result it remains to show that $l(\alpha)$ is tame. The property of definable functions is preserved by operations including the sum, composition by a linear operator, derivation and canonical projection ([34], [14]). Since T is a tame mapping, $I + M^{-1/2}TM^{-1/2}$ is tame as well as its graph. Here I is identity. Then the resolvent $J_{M^{-1/2}TM^{-1/2}} = (I + M^{-1/2}TM^{-1/2})^{-1}$ which is defined by the inverse of the same graph is tame ([18]) and single-valued. By the stability of the sum and composition by linear operator, we obtain that $l(\alpha)$ is tame. □

A.8 Proof of Proposition 4.3

Proof. Let $p = J_T^V(z)$. Since resolvent operator is non-expansive, $\|p\| = \|J_T^V(z)\| \le \|z\| + \|J_T^V(0)\|$. By duality, optimal α^* will satisfy $\alpha^* = u^*(p - z)$. Then,

$$\begin{aligned} |\alpha^*| &= |u^*(p - z)| \\ &\le \|u\|(2\|z\| + \|J_T^V(0)\|)\,. \quad \square \end{aligned} \tag{101}$$

A.9 Proof of Proposition 4.5

Proof. $\mathcal{L}(\alpha)$ is as defined in Proposition 4.4. Substitute $\alpha = \xi + U^*V^{-1}Bz$ in $\mathcal{L}(\alpha)$. We obtain $\mathcal{J}(\xi) = \mathcal{L}(\alpha)$. It yields $\alpha^* = \xi^* + U^*V^{-1}Bz$. In (24), we do the same substitution.

$$\begin{aligned} \hat{z} &= J_T^M(\breve{z} \mp M^{-1}U\alpha^*) \\ &= J_A^M(\breve{z} \mp M^{-1}U\xi^* \mp M^{-1}UU^*V^{-1}Bz) \\ &= J_A^M(z - M^{-1}MV^{-1}Bz \mp M^{-1}UU^*V^{-1}Bz \mp M^{-1}U\xi^*) \\ &= J_A^M(z - M^{-1}\underbrace{(M \pm UU^*)}_{=V}V^{-1}Bz \mp M^{-1}U\xi^*) \\ &= J_A^M(z - M^{-1}Bz \mp M^{-1}U\xi^*)\,. \end{aligned} \tag{102}$$

Since $\mathcal{L}(\alpha)$ is Lipschitz with constant $1 + ||M^{-1/2}U||^2$ and strongly monotone, the composition of translation preserves the same properties. □

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