Global non-asymptotic super-linear convergence rates of regularized proximal quasi-Newton methods on non-smooth composite problems

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Abstract

In this paper, we propose two regularized proximal quasi-Newton methods with symmetric rank-1 update of the metric (SR1 quasi-Newton) to solve non-smooth convex additive composite problems. Both algorithms avoid using line search or other trust region strategies. For each of them, we prove a super-linear convergence rate that is independent of the initialization of the algorithm. The cubic regularized method achieves a rate of order $\left(\frac{C}{N^{1/2}}\right)^{N/2}$, where N is the number of iterations and C is some constant, and the other gradient regularized method shows a rate of the order $\left(\frac{C}{N^{1/4}}\right)^{N/2}$. To the best of our knowledge, these are the first global non-asymptotic super-linear convergence rates for regularized quasi-Newton methods and regularized proximal quasi-Newton methods. The theoretical properties are also demonstrated in two applications from machine learning.

1 Introduction

Newton-type methods have been studied extensively over decades due to their fast convergence. However, the complexity of the computational cost of Newton's method per iteration is cubic, making it intractable for large-scale applications. Therefore, concurrently, quasi-Newton methods have been developed to avoid the explicit computation of the Hessian (second derivative) of the objective [14]. The idea of quasi-Newton methods is to approximate the Hessian by a matrix that is generated from first-order information, for example, using the difference of the gradient of the objective at nearby points. Based on this idea, numerous variants have been developed, including BFGS [6], SR1 [12, 5], DFP [12, 19]. Classically, the convergence guarantee of both, Newton's method and quasi-Newton methods, are local, i.e., they require the starting point to be sufficiently close to an optimal point, unless globalization strategies such as trust region or line search are applied [11]. This picture has changed thanks to the pioneering work of Nesterov and Polyak [34]. They propose a cubic regularization strategy for Newton's method that guarantees global convergence without line search. More recently, following the same goal, gradient regularization was introduced to Newton's method [33]. While these regularization strategies stabilize the algorithm and allow for global convergence, they do not remedy the enormous computational cost of involving the Hessian of the objective in the update step. Therefore, it is natural to take this as inspiration for the design of regularized quasi-Newton methods that converge globally and are also applicable to large-scale problems. However, even for strongly convex functions, the question remains:

how can we design a globally convergent cubic- (or gradient-) regularized quasi-Newton method following the spirit of the pioneering works [34, 33].

A striking property of quasi-Newton-type methods is their super-linear rate of convergence [6, 7, 15, 18, 20], which however in most cases can only be proved locally or are only asymptotic rates. Recently, [38] provides the first non-asymptotic super-linear convergence rate for greedy quasi-Newton methods, which is refined to explicit rates for a restricted Broyden family of quasi-Newton methods in [40, 25]. In [44], the first non-asymptotic explicit super-linear convergence rate for the SR1 method is proved. While these rates are non-asymptotic, they are still local, meaning that the initialization is required to be sufficiently close to an optimal point. In this sense, a local region around an optimal point is defined in which the super-linear rate of convergence can be observed. This raises the natural question of

whether it is possible to design quasi-Newton methods with global convergence and global non-asymptotic super-linear convergence rates.

In this paper, we give an affirmative answer to both questions above by proposing cubic- and gradient-regularized quasi-Newton methods. In Table 1, we list the main features and limitations of the state-of-the-art quasi-Newton methods and arrange our proposed regularized quasi-Newton methods in this context, which we explain in more detail in the following. Unlike other works mentioned in Table 1 and Table 2, our methods attain a non-asymptotic super-linear rate of convergence that is independent of the initialization and without using expensive line search or trust region sub-routines. The globalization is achieved via regularization and the regime of super-linear convergence is encoded in the number of iterations instead of a hard-to-estimate local region around an optimal point. Instead of assuming the Dennis–Moré criterion for convergence [10, 28], which is usually difficult to validate, we rely on Lipschitz continuity of the Hessian instead.

State-of-the-art quasi-Newton-type methods with super-linear rate of convergence		
Scheme	Rate	Local region of super-linear convergence
DFP [40]	$\left(\frac{nL^2}{\mu^2 N}\right)^{N/2}$	$\frac{L_H}{\mu^{3/2}}\lambda_f(x_0) \le \frac{\log\frac{3}{2}\mu}{4L}$
BFGS [40]	$\left(\frac{nL}{\mu N}\right)^{N/2}$	$\frac{L_H}{\mu^{3/2}}\lambda_f(x_0) \le \frac{\log\frac{3}{2}\mu}{4L}$
DFP [25]	$\left(\frac{1}{N}\right)^{N/2}$	$\frac{L_H}{\mu^{3/2}}\|\nabla^2 f(x_*)^{1/2}(x_0 - x_*)\| \le \min\{\frac{1}{120}, \frac{1}{7\sqrt{n}}\}$
BFGS [25]	$\left(\frac{1}{N}\right)^{N/2}$	$\frac{L_H}{\mu^{3/2}}\|\nabla^2 f(x_*)^{1/2}(x_0 - x_*)\| \le \min\{\frac{1}{50}, \frac{1}{7\sqrt{n}}\}$
SR1 [44]	$\left(\frac{2n\log 4\varkappa}{N}\right)^{N/2}$	$\frac{L_H}{\mu^{3/2}}\lambda_f(x_0) \le \frac{\log\frac{3}{2}}{4\sqrt{\frac{3}{2}}}\max\{\frac{1}{2\varkappa}, \frac{1}{3n\log 4\varkappa+3}\}$
Cubic SR1 PQN (Alg 1)	$\left(\frac{C}{N^{1/2}}\right)^{N/2}$	global
Grad SR1 PQN (Alg 2)	$\left(\frac{C_{\text{grad}}}{N^{1/4}}\right)^{N/2}$	global

Table 1: We compare our methods with classical quasi-Newton methods on the smooth problem $\min_{x\in\mathbb{R}^n} f$ where f has L-Lipschitz gradient and is μ-strongly convex. Here, we rewrite the results in our notations. N is the number of iterations, x_* denotes the optimal point, $\lambda_f(x_0) := \|\nabla f(x_0)\|_{\nabla^2 f(x_0)^{-1}}$ and $\varkappa := \frac{L}{\mu}$. C, C_{grad} are constants (see Theorems 3.1 and 3.4).

Comparison of recent works on convergence rates of quasi-Newton methods				
Results	Non-asymptotic	Explicit	Global	Without Line search and trust region strategy
Broyden family [38, 40, 25]	Yes	Yes	No	Yes
SR1 [44]	Yes	Yes	No	Yes
BFGS [24]	Yes	Yes	Yes	No (exact line search)
BFGS [23, 37]	Yes	Yes	Yes	No (inexact line search)
BFGS [22]	Yes	Yes	Yes	Yes (online learning strategy)
Our Alg 1 and 2	Yes	Yes	Yes	Yes (cubic- or grad-regularization)

Table 2: Works related to ours that have achieved local or global convergence with asymptotic or non-asymptotic explicit super-linear rate of convergence.

The analysis of our algorithms is built on a novel Lyapunov-type potential function that is in fact simpler than those employed in related work. We summarize recently used potential functions in Table 3. While our algorithms and analysis already demonstrate a significant contribution for smooth optimization problems, furthermore, we consider non-smooth additive composite problems for which proximal gradient type algorithms are the standard choice for the large-scale regime. Such problems frequently appear in several areas in machine learning, computer vision, image or signal processing, and statistics. In this structured non-smooth case, the quasi-Newton metric is constructed using the smooth part of the objective function, whereas the non-smooth part of the objective is handled via a proximal step that is computed in the same metric.

Potential function	
Broyden Family [40]	$\sigma(A,G) = \mathrm{tr}\,(A^{-1}(G-A))$ for $A \preceq G$
Broyden Family [40, 8]	$\sigma(A,G) = \mathrm{tr}\,(A^{-1}(G-A)) - \log\det(A^{-1}G)$ for $A \preceq G$
Broyden Family [25, 7]	Frobenius-norm involved potential functions
SR1[44], Broyden Family [39]	$\sigma(A,G) = \log\det(A^{-1}G)$ and $\mathrm{tr}\,(G-A)$ for $\sigma(A,G) = A \preceq G$
Greedy SR1[30]	$\sigma(A,G) = \mathrm{tr}\,(G-A)$ for $A \preceq G$
Our Cubic/Grad SR1 PQN	$V(G) = \mathrm{tr}\,G$

Table 3: We list the potential functions introduced or used in the literature to prove super-linear convergence rates of quasi-Newton methods along with our potential function. Here $\mathrm{tr}\,G$ denotes the trace of the matrix G and $\det G$ denotes the determinant of the matrix G.

2 Related Works

Regularized Newton method. In the classical literature on Newton's method [35], local super-linear and quadratic rates of convergence for the pure Newton's method can be proved. In order to achieve global convergence, certain globalization strategies must be used such as line search or trust region approaches. Sometimes global convergence can be obtained while maintaining the local fast convergence [13]. There are other regularization strategies that add a positive definite matrix to the Hessian (like in the Levenberg–Marquardt method [29, 32]) in order to stabilize Newton's method and to obtain global convergence thanks to having a positive definite metric. However, in general, this changes the Hessian and global convergence is obtained at the cost of loosing the local fast convergence rates. The pioneering work by Nesterov and Polyak [34] has introduced cubic regularization as another globalization strategy that avoids line search and at the same time shows the standard fast local convergence rates of the pure Newton's method. In order to remedy the computational cost of cubic regularization, in [33, 17] a gradient regularization strategy was introduced that allows for global convergence with a super-linear rate of convergence for strongly convex problems. *These works have fundamentally motivated the present paper and the development of our cubic and gradient regularized quasi-Newton methods that combines the best of both worlds: a global explicit super-linear rate of convergence while preserving a low computational cost per iteration by using only first-order information.*

Quasi-Newton methods. Rodomanov and Nesterov [38] obtained the first local explicit super-linear convergence rate for greedy quasi-Newton methods and [44] obtained the first local explicit super-linear convergence rate for the SR1 quasi-Newton method, which both inspired our development in this paper. In subsequent works, [40, 25] proved the local non-asymptotic super-linear convergence rate for the classical Broyden family of quasi-Newton methods (Table 1 and Table 2). However, all results above require the starting point to lie close to the optimal point to achieve the super-linear rate of convergence, unless additionally a line search strategy is used. Therefore, it is worth looking for a global non-asymptotic super-linear convergence rate. The recent paper [22] showed a global non-asymptotic super-linear convergence rate by using an online learning strategy. For BFGS with an exact line search strategy, [24] obtain a global non-asymptotic convergence rate. The rate is linear at the beginning and super-linear when the iterates approach the local super-linear convergence region of the BFGS method. Later, [37, 23] achieve similar results with an inexact line search strategy, independently. *In our paper, we combine the idea of regularization (cubic, gradient) and SR1 method and propose methods that are shown to converge globally with a global non-asymptotic super-linear rate without the help of globalization strategies (line search, trust region method).*

There are a few more works, appearing under similar names like 'regularized Newton-type method' [9, 26, 3, 4, 31, 21]. Since they do not consider non-asymptotic super-linear convergence rates and are using line search or trust region strategies, we do not describe them in more detail here.

Potential functions for convergence analysis. To obtain the non-asymptotic super-linear convergence, potential functions are widely used. We would like to mention several seminal papers on selecting potential functions (listed in Table 3). [7] used the Frobenius-norm function for studying the Broyden family of quasi-Newton methods. Later, an important contribution was made by Byrd, Liu, Nocedal and Yuan in [8], who introduced the useful potential function $\sigma(A, G) = \operatorname{tr}(A^{-1}(G-A)) - \log\det A^{-1}G$. [40] used the first part of this function $\sigma(A, G) = \operatorname{tr}(A^{-1}(G-A))$ to show the convergence of the Broyden family of quasi-Newton methods on quadratic functions. A similar trace function $\operatorname{tr}(G - A)$ was first introduced in [30]. In [44], the authors also used the function $\operatorname{tr}(G-A)$ to study the quadratic case. However, we must emphasize that for their proofs, the existence of A is crucial and necessary. That is the reason why we regard our potential function, despite its apparent similarity to the previous ones,, as a novel contribution. *We propose the potential function $V(G) = \operatorname{tr} G$ which is much simpler than other potential functions while it also eases the proof of our non-asymptotic global super-linear convergence rate. Moreover, it provides a simple criterion for restarting the quasi-Newton method to guarantee the global convergence of our gradient regularized Algorithm 2 (and its variant in Algorithm 3).*

Proximal quasi-Newton and Newton-type methods. One of the earliest work on proximal quasi-Newton methods is [10], where they adopted line search to guarantee global convergence and assume the Dennis–Moré criterion to obtain asymptotic super-linear convergence. [28] also demonstrated an asymptotic super-linear local convergence rate of the proximal Newton and a proximal quasi-Newton method under the same condition. We believe that the Dennis–Moré criterion is very restrictive and hard to check. Therefore, we rather use Lipschitz continuity of the Hessian instead. Later, [41] showed that by a prox-parameter update mechanism, rather than a line search, they can derive a sub-linear global complexity. However, to the best of our knowledge, there is no result on non-asymptotic explicit super-linear convergence rates for a proximal quasi-Newton method. While the works discussed so far focus on the convergence rate, general convergence of so-called variable metric proximal gradient methods was also studied intensively. A crucial topic, which is mostly ignored, is the practical computation of the proximal mapping with respect to the metric that is induced by the quasi-Newton metric. Usually, simple proximal mappings in the standard metric become hard to solve when the metric is changed. There are a few works that are dedicated to this topic, proposing a proximal calculus tailored to exactly this situation. This was initiated in [1] and consolidated in [2], where the interested reader finds an extensive literature review around this topic. Several follow up works have extended and improved these results [41, 27, 2, 43, 26].

3 Problem Setup and Main Results

In this section, we present our notation, the considered problem setup with all standing assumptions, our proposed regularized proximal SR1-quasi-Newton algorithms and the main convergence results. We postpone the technical parts of the convergence analysis to Section 4.

3.1 Notation

We first introduce our notations. We denote $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$. $\mathbb{R}^n$ is an Euclidean vector space of dimension n, which is equipped with the standard Euclidean norm $\|u\| := \sqrt{u^\top u}$, for any $u \in \mathbb{R}^n$. Given a matrix $A \in \mathbb{R}^{n\times n}$, $\|A\|$ denotes the matrix norm induced by the Euclidean vector norm. We write $\operatorname{tr} A$ for the trace of the matrix $A \in \mathbb{R}^{n\times n}$. Moreover, I denotes the identity matrix in $\mathbb{R}^n$. We adopt the standard definition of the Loewner partial order of symmetric positive semi-definite matrices. Let A and B be two symmetric matrices, we say $A \preceq B$ (or $A \prec B$) if and only if for any $u \in \mathbb{R}^n$, we have $u^\top(B - A)u \geq 0$ (or $u^\top(B - A)u > 0$, respectively).

3.2 Problem set up

In the whole paper, we consider the following optimization problem:

$$\min_{x\in\mathbb{R}^n} F(x) := g(x) + f(x)\,, \tag{1}$$

where we make the following assumptions:

Assumption 1. *1. $F = g + f$ is bounded from below;*

2. $g\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ is proper, lower semi-continuous (lsc), convex;

3. $f\colon \mathbb{R}^n \to \mathbb{R}$ is twice differentiable with L-Lipschitz gradient, L_H-Lipschitz Hessian and f is μ-strongly convex.

3.3 Algorithms and main results

Let us first recap the classical symmetric rank-1 quasi-Newton (SR1) update $G_+ := \mathrm{SR1}(A, G, u)$. Given two $n \times n$ positive definite matrices A and G such that $0 \prec A \preceq G$ and $u \in \mathbb{R}^n$, we define the SR1 update as follows:

$$\mathrm{SR1}(A, G, u) := \begin{cases} G\,, & \text{if } (G-A)u = 0\,; \\ G - \frac{(G-A)uu^\top(G-A)}{u^\top(G-A)u}\,, & \text{otherwise}\,. \end{cases} \tag{2}$$

In order to solve the problem in (1), we propose two regularized proximal SR1 quasi-Newton methods: Algorithms 1 and 2, where we present a variant of the latter as Algorithm 3 in the appendix.

Algorithm 1 is inspired by the cubic regularized Newton method from [34], which is proved to converge globally. Step 1 of Algorithm 1 augments the classical quasi-Newton update step with a cubic regularization term that is scaled with the Lipschitz constant L_H of the Hessian of f. Step 2 defines several variables, including the stationarity measure $F'(x_{k+1})$ that is also used to terminate the algorithm in Step 6. In Step 3, we design a new correction strategy such that $\tilde{G}_k \succeq J_k$ (discussed in detail in Section 4), where J_k is defined in Step 4. It is crucial to observe that this is only a theoretical quantity for the analysis of the algorithm. In practice, J_k is only evaluated in the product with $u_k = x_{k+1} - x_k$ for which it simplifies to $J_k u_k = \nabla f(x_{k+1}) - \nabla f(x_k)$. Finally, Step 5 computes the SR1 update of the metric from (2), which for $(\tilde{G}_k - J_k)u_k \neq 0$ (namely $F'(x_{k+1}) \neq 0$), becomes

$$G_{k+1} = \mathrm{SR1}(J_k, \tilde{G}_k, u_k) = G_k + \lambda_k I - \frac{((G_k + \lambda_k I)u_k - y_k)\,((G_k + \lambda_k I)u_k - y_k)^\top}{u_k^\top((G_k + \lambda_k I)u_k - y_k)}\,,$$

where $y_k := \nabla f(x_{k+1}) - \nabla f(x_k)$.

Algorithm 1 Cubic SR1 PQN

Require: x_0, $G_0 = LI$, $r_{-1} = 0$.

Update for $k = 0, \cdots, N$:

1. **Update**

$$x_{k+1} = \operatorname*{argmin}_{x\in\mathbb{R}^n} \{g(x) + \langle \nabla f(x_k), x - x_k\rangle + \frac{1}{2}\|x - x_k\|^2_{G_k + L_H r_{k-1} I} + \frac{L_H}{3}\|x - x_k\|^3\}. \tag{3}$$

2. **Set** $u_k = x_{k+1} - x_k$, $r_k = \|u_k\|$, $\lambda_k = L_H r_{k-1} + L_H r_k$ and

$$F'(x_{k+1}) = \nabla f(x_{k+1}) - \nabla f(x_k) - \tilde{G}_k u_k\,.$$

3. **Compute** $\tilde{G}_k = G_k + \lambda_k I$.
4. **Denote but not use explicitly** $J_k := \int_0^1 \nabla^2 f(x_k + t u_k) dt$.
5. **Compute** $G_{k+1} = \mathrm{SR1}(J_k, \tilde{G}_k, u_k)$.
6. **If** $\|F'(x_{k+1})\| = 0$: Terminate.

End

For the cubic regularized proximal quasi-Newton method in Algorithm 1, we obtain the following global convergence with a non-asymptotic super-linear rate.

Theorem 3.1 (Main Theorem for Algorithm 1)**.** *For any initialization $x_0 \in \mathbb{R}^n$ and any $N \in \mathbb{N}$, Cubic SR1 PQN(Algorithm 1) has a global convergence with the rate:*

$$\|F'(x_N)\| \leq \left(\frac{C}{N^{1/2}}\right)^{N/2} \|F'(x_0)\|, \tag{4}$$

where $C := (2nL_H C_0 + nL)/\mu$ and $C_0 := \sqrt{\frac{2(F(x_0)-\inf F)}{\mu}}$.

Proof. See Section 4.2. □

Remark 3.2. *In fact, (4) indicates that the super-linear rate is only attained after a certain number of iterations $\bar{N}$ such that $\bar{N} \geq C^2$, where $C = O(\frac{n}{\mu^{3/2}})$. $\bar{N}$ is comparable to $O(n^2)$ which is an important price to pay for globality. However, even for an initialization close to the optimal point, the super-linear rate of [38, 44] is attained after $\bar{N} = O(n)$ iterations. While the super-linear rate of [25] is attained immediately, the radius of their local region of convergence is as small as $O(1/\sqrt{n})$ when n is large. Besides, the local region criterion of [25] requires the knowledge of the optimal point x_*, making this criterion useless.*

Remark 3.3. *As pointed out in [17], $\|F'(x)\|$ can approach zero since $F'(x)$ is a very special way of selecting subgradient of a possibly non-smooth function $F(x)$.*

When $g \equiv 0$, Algorithm 1 is a cubic regularized SR1 quasi-Newton method for solving a smooth optimization problem, and is also a novel contribution. In this case, the global convergence rate in (4) can be stated explicitly in term of the gradient of the (smooth) objective

$$\|\nabla f(x_N)\| \leq \left(\frac{C}{N^{1/2}}\right)^{N/2} \|\nabla f(x_0)\|.$$

Despite having the favorable rate of convergence, the computational complexity per iteration can be costly, since the cubic regularized subproblem that needs to be minimized in Step 3 usually does not have a closed form solution. While the same is true also for cubic regularized Newton's method [34], the additional cost casts our algorithm inpractical for large-scale applications. In our implementation, we have adopted the strategy from [34, Section 5], which proposes a reformulation into a convex one-dimensional subproblem and the usage of a bisectioning method. To remedy the high computational cost caused by the cubic regularization term, inspired by [33, 17], we provide two proximal quasi-Newton methods with gradient regularization (Algorithm 2 and 3), which come with the same theoretical guarantees while maintaining a low computational cost per iteration.

Algorithm 2 is derived from Algorithm 1. Step 1 of Algorithm 2 equips the classical quasi-Newton update with a gradient regularization term instead of a cubic regularization. Step 2 defines J_k which has the same exclusively theoretical purpose as in Algorithm 1. In Step 3, we define several variables, including $F'(x_{k+1})$ which is used to generate the correction factor λ_{k+1} and terminate the algorithm in Step 5. Step 4 is a new correction strategy. First, we compute $\hat{G}_{k+1}$ with a scaling parameter $1 + \lambda_{k+1}$. Second, we set a restarting criterion based on the trace of $\hat{G}_{k+1}$. If the trace of $\hat{G}_{k+1}$ is bounded by $n\bar{\kappa}$ where $\bar{\kappa} \geq L$ is a hyper-parameter set at the beginning, we update the gradient regularized quasi-Newton metric $\tilde{G}_{k+1}$, otherwise, we restart the update of the quasi-Newton metric by setting $\tilde{G}_{k+1} = LI$.

Algorithm 2 Grad SR1 PQN

Require: x_0, $G_0 = LI$, $\lambda_0 = 0$, $\tilde{G}_0 = G_0(1+\lambda_0)$, $\bar{\kappa} \geq L$.

Update for $k = 0, \cdots, N$:

1. **Update**

$$x_{k+1} = \operatorname*{argmin}_{x\in\mathbb{R}^n} g(x) + \langle \nabla f(x_k), x - x_k\rangle + \frac{1}{2}\|x - x_k\|^2_{\tilde{G}_k}. \tag{5}$$

2. **Denote but not use explicitly** $J_k := \int_0^1 \nabla^2 f(x_k + tu_k)dt$.
3. **Compute** $u_k = x_{k+1} - x_k$, $r_k = \|u_k\|$, and

$$\begin{aligned} F'(x_{k+1}) &= \nabla f(x_{k+1}) - \nabla f(x_k) - \tilde{G}_k u_k\,, \\ G_{k+1} &= \mathrm{SR1}(J_k, \tilde{G}_k, u_k)\,, \\ \lambda_{k+1} &= \frac{1}{\mu}\left(\sqrt{L_H\|F'(x_{k+1})\|} + L_H r_k\right). \end{aligned}$$

4. **Update** $\tilde{G}_{k+1}$**:** compute $\hat{G}_{k+1} = (1+\lambda_{k+1})G_{k+1}$ (**Correction step**).
   If $\operatorname{tr}\hat{G}_{k+1} \leq n\bar{\kappa}$**:** we set $\tilde{G}_{k+1} = \hat{G}_{k+1}$.
   Othewise: we set $\tilde{G}_{k+1} = LI$ (**Restart step**).
5. **If** $\|F'(x_{k+1})\| = 0$: Terminate.

End

In Algorithm 2, λ_k relies on the subgradient $F'(x_k)$ of F at x_k which is different from what we did in Algorithm 1 and makes (5) easier to solve compared to the cubic regularization. The gradient regularization, despite being inspired by [33], is tailored to the non-smooth composite problem. The restarting strategy in Step 4 from Algorithm 2, respectively, ensures that for any $k \geq 1$, the metric $\tilde{G}_k$ is bounded, namely, $\tilde{G}_k \preceq \operatorname{tr}\tilde{G}_k I \preceq n\bar{\kappa} I$ where $\bar{\kappa} \geq L$.

For gradient regularized proximal quasi-Newton methods, we obtain global convergence with the following non-asymptotic super-linear rate.

Theorem 3.4 (Main Theorem for Algorithm 2)**.** *For any initialization $x_0 \in \mathbb{R}^n$ and any $N \in \mathbb{N}$, Grad SR1 PQN (Algorithm 2) has a global convergence with the rate:*

$$\|F'(x_N)\| \leq \left(\frac{C_{\mathrm{grad}}}{N^{1/4}}\right)^{N/2} \|F'(x_0)\|\,, \tag{6}$$

where $C_{\mathrm{grad}} := (n\bar{\kappa}\Theta + nL)/\mu$, $\Theta = \frac{1}{\mu}\left(L_H C_0 + \sqrt{L_H(L+n\bar{\kappa})C_0}\right)$ *and* $C_0 := \sqrt{\frac{2(F(x_0)-\inf F)}{\mu}}$.

Proof. See Section 4.3. □

Remark 3.3 also applies for Theorem 3.4.

Remark 3.5. *In fact, (6) indicates that the super-linear rate is only attained after $N \geq C^4_{\mathrm{grad}}$ number of iterations, where $C_{\mathrm{grad}} = O(\frac{n}{\mu^{5/2}})$.*

The rate for gradient regularization is worse than for cubic regularization, since the super-linear rate is guaranteed to occur only after $N \geq C^4_{\mathrm{grad}}$ number of iterations. However, since the cubic regularization is no longer needed, the computational cost of the update step is significantly reduced. When $g \equiv 0$, the convergence rate in (6) becomes

$$\|\nabla f(x_N)\| \leq \left(\frac{C_{\mathrm{grad}}}{N^{1/4}}\right)^{N/2} \|\nabla f(x_0)\|\,,$$

and, in this smooth case, the update step in Step 1 is as follows:

$$x_{k+1} = x_k - \tilde{G}_k^{-1}\nabla f(x_k)\,, \tag{7}$$

where the computational cost is, in fact, $O(n^2)$, since the use of $\tilde{G}_k = (1+\lambda_k)G_k$ in Step 4 enables the application of the Sherman–Morrison–Woodbury formula.

When $g \not\equiv 0$, the update step has the same computational cost as that of a classical proximal quasi-Newton method and is much more tractable than that of Algorithm 1 which uses cubic regularization.

Algorithm 3. For the sake of completeness, in Section A in the appendix, we present a variant of Algorithm 2, which comes with the same convergence guarantees under a slight modification of the correction step, which is closer to the existing related works (e.g. [26]) and has a smaller constant $C_{\text{grad}} = O(\frac{n}{\mu^{3/2}})$ which shows a better dependence on the conditioning, however the computational cost of the update step in the smooth case is larger than that of Algorithm 2.

4 Convergence Analysis

In this section, without loss of generality, we assume that $\|F'(x_k)\| > 0$ for any $k \in \mathbb{N}$, since, otherwise, our algorithms terminate after a finite number of steps and our non-asymptotic rates would hold until then.

Recall that throughout the whole convergence analysis, we assume that Assumption 1 holds and that all variables are defined either as in Algorithm 1 or 2. The convergence analysis of Algorithm 3 can also be found in the appendix.

4.1 Preliminary

First, we need several important properties of $J_k \coloneqq \int_0^1 \nabla^2 f(x_k + tu_k)dt$, which is the same for both algorithms.

Lemma 4.1. *For each $k \in \mathbb{N}$, we have $J_k \preceq LI$.*

Proof. Since f has L-Lipschitz gradient, then, for any x, we have $\nabla^2 f(x) \preceq LI$. Therefore, due to the definition of J_k, we have $J_k \preceq LI$. ☐

Lemma 4.2. *For each $k \in \mathbb{N}$, we have*

$$\mu I \preceq J_k \preceq J_{k-1} + L_H(r_k + r_{k-1})I\,. \tag{8}$$

Proof. Since $f(x)$ is μ-strongly convex, the left first is trivial. Thus, we focus on the second inequality. The proof is similar with the one of [38, Lemma 4.2] and [44, Lemma 5]. The assumption that the Hessian of f is L_H-Lipschitz continuous means that there exists $L_H > 0$ such that for any $x, y \in \mathbb{R}^n$, we have

$$\|\nabla^2 f(x) - \nabla^2 f(y)\| \leq L_H\|x - y\|\,. \tag{9}$$

For any $t \in [0, 1]$ and $k \in \mathbb{N}$, letting $x = x_k + tu_k$ and $y = x_k - (1-t)u_{k-1}$, we have that

$$\|\nabla^2 f(x_k + tu_k) - \nabla^2 f(x_k - (1-t)u_k)\| \leq L_H\|tu_k + (1-t)u_{k-1}\|\,. \tag{10}$$

Then, by the definition of J_k, we have

$$\begin{aligned} J_k - J_{k-1} &= \int_0^1 \nabla^2 f(x_k + tu_k)dt - \int_0^1 \nabla^2 f(x_{k-1} + tu_{k-1})dt \\ &= \int_0^1 \nabla^2 f(x_k + tu_k)dt - \int_0^1 \nabla^2 f(x_k - (1-t)u_{k-1})dt \\ &\preceq \int_0^1 \|\nabla^2 f(x_k + tu_k) - \nabla^2 f(x_k - (1-t)u_{k-1})\|dt \cdot I \\ &\preceq \int_0^1 L_H\|tu_k + (1-t)u_{k-1}\|dt \cdot I \\ &\preceq L_H(r_k + r_{k-1})I\,, \end{aligned} \tag{11}$$

where the first inequality holds because of the definition of the induced matrix norm, the second inequality holds due to (10) and the last one uses the triangle inequality. ☐

Then, we recall an important property of SR1 method, which shows that the update of SR1 method preserves the partial order of matrices.

Lemma 4.3. *For any two symmetric positive definite matrices $A \in \mathbb{R}^{n\times n}$ and $G \in \mathbb{R}^{n\times n}$ with $0 \preceq A \preceq G$ and any $u \in \mathbb{R}^n$, we observe the following for $G_+ = SR1(A, G, u)$:*

$$A \preceq G_+ \preceq G\,. \tag{12}$$

Proof. We recall the short proof from [38, Part of Lemma 2.2] here for readers' convenience. If $(G-A)u = 0$, the result is obvious. If $(G-A)u \neq 0$, we conclude the statement as follows:

$$G_+ - A = G - A - \frac{(G-A)uu^\top(G-A)}{u^\top(G-A)u} = (G-A)^{1/2}(1 - \frac{\tilde{u}\tilde{u}^\top}{\tilde{u}^\top\tilde{u}})(G-A)^{1/2} \succeq 0\,, \tag{13}$$

where $\tilde{u} = (G-A)^{1/2}u$. □

As motivated in the introduction, for any symmetric positive definite matrix G, we introduce a new potential function that is key to our convergence analysis:

$$V(G) \coloneqq \operatorname{tr} G\,. \tag{14}$$

Moreover, for any $G \succeq 0$ with $V(G) \geq 0$, given two matrices A and G satisfying $A \preceq G$, additionally, we introduce the following measurement function:

$$\nu(A, G, u) = \begin{cases} 0\,, & \text{if } (G-A)u = 0\,; \\ \frac{u^\top(G-A)(G-A)u}{u^\top(G-A)u}\,, & \text{otherwise}\,. \end{cases} \tag{15}$$

Using the potential function $V(G)$ and $\nu(A, G, u)$, we deduce the following lemma showing that the SR1 update leads to a better approximation of A.

Lemma 4.4. *Consider two symmetric positive definite matrices $A \in \mathbb{R}^{n\times n}$ and $G \in \mathbb{R}^{n\times n}$ with $0 \prec A \preceq G$ and any $u \in \mathbb{R}^n$. Let $G_+ = SR1(A, G, u)$. Then, we have*

$$V(G) - V(G_+) = \nu(A, G, u)\,, \tag{16}$$

and

$$\nu(A, G, u) \geq \frac{\|(G-A)u\|^2}{u^\top Gu}\,. \tag{17}$$

Proof. The result is trivial for the case $(G-A)u = 0$, since this yields $G_+ = G$. Therefore, let $(G-A)u \neq 0$. A simple calculation shows

$$V(G) - V(G_+) = \operatorname{tr}(G - G_+) = \operatorname{tr}\left(\frac{(G-A)uu^\top(G-A)^\top}{u^\top(G-A)u}\right) = \left(\frac{\|(G-A)u\|^2}{u^\top(G-A)u}\right)\,,$$

in which the last term can be bounded from below as claimed thanks to the ordering $G \succeq A \succ 0$. □

4.2 Convergence analysis of Algorithm 1

Let us focus on Algorithm 1. The optimality condition of the update step in (3) is

$$\partial g(x_{k+1}) + \nabla f(x_k) + (G_k + L_H r_{k-1})(x_{k+1} - x_k) + L_H\|x_{k+1} - x_k\|(x_{k+1} - x_k) \ni 0\,, \tag{18}$$

which, using our notations, simplifies to

$$\partial g(x_{k+1}) + \nabla f(x_k) + (G_k + \lambda_k)(x_{k+1} - x_k) \ni 0\,. \tag{19}$$

We denote $v_{k+1} \coloneqq -\nabla f(x_k) - (G_k + \lambda_k)(x_{k+1} - x_k)$ and $F'(x_k) \coloneqq v_k + \nabla f(x_k)$ and obtain $F'(x_k) \in \partial F(x_k)$.

The following lemma is important for the correction step from G_k to $\tilde{G}_k$, showing that for each k, we have $\tilde{G}_k \succeq J_k$.

Lemma 4.5. *For any $k \in \mathbb{N}$, we have that $\tilde{G}_k = G_k + L_H(r_k + r_{k-1})I \succeq G_{k+1} \succeq J_k$.*

Proof. We prove the result by induction. For $k = 0$, by definition, we have $J_0 \preceq LI \preceq \tilde{G}_0$, and using Lemma 4.3, we have $J_0 \preceq G_1 = \mathrm{SR1}(J_0, \tilde{G}_0, u_0) \preceq \tilde{G}_0$. Now, we assume that $J_{k-1} \preceq G_k \preceq \tilde{G}_{k-1}$ holds for $k-1$. Then, for k, we have

$$J_k \preceq J_{k-1} + L_H(r_{k-1} + r_k)I \preceq G_k + L_H(r_k + r_{k-1})I = \tilde{G}_k \,, \tag{20}$$

where the first inequality holds due to Lemma 4.2, which shows that $0 \preceq J_k \preceq \tilde{G}_k$ and we conclude the induction by Lemma 4.3:

$$J_k \preceq G_{k+1} = \mathrm{SR1}(J_k, \tilde{G}_k, u_k) \preceq \tilde{G}_k \,. \qquad \square$$

Now, we study the descent lemma of function values.

Lemma 4.6. *For any $k \in \mathbb{N}$, we have*

$$F(x_{k+1}) - F(x_k) \le -\frac{\mu}{2} r_k^2 \,. \tag{21}$$

Proof. We start with the strong convexity of the smooth function f: for any x and y, we have

$$f(x) - f(y) \le \langle \nabla f(x), x - y \rangle - \frac{\mu}{2}\|x - y\|^2 \,. \tag{22}$$

Letting $x = x_{k+1}$ and $y = x_k$, we have

$$\begin{aligned} f(x_{k+1}) - f(x_k) &\le \langle \nabla f(x_{k+1}), x_{k+1} - x_k \rangle - \frac{\mu}{2}\|x_{k+1} - x_k\|^2 \\ &= \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \langle \nabla f(x_{k+1}) - \nabla f(x_k), x_{k+1} - x_k \rangle - \frac{\mu}{2}\|x_{k+1} - x_k\|^2 \\ &= \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \langle J_k(x_{k+1} - x_k), x_{k+1} - x_k \rangle - \frac{\mu}{2}\|x_{k+1} - x_k\|^2 \,, \end{aligned} \tag{23}$$

where the last equality holds since $J_k u_k = \int_0^1 \nabla^2 f(x_k + t u_k) u_k dt = \nabla f(x_{k+1}) - \nabla f(x_k)$. According to Lemma 4.2, we have $\langle J_k u_k, u_k \rangle \le \langle J_{k-1} u_k + L_H(r_k + r_{k-1}) u_k, u_k \rangle$. Thus, (23) implies that

$$\begin{aligned} f(x_{k+1}) - f(x_k) \le \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \langle J_{k-1}(x_{k+1} - x_k), x_{k+1} - x_k \rangle \\ + L_H(r_k + r_{k-1})\|x_{k+1} - x_k\|^2 - \frac{\mu}{2}\|x_{k+1} - x_k\|^2 \,. \end{aligned} \tag{24}$$

By Lemma 4.5, we have $G_k \succeq J_{k-1}$ and thus $\langle G_k u_k, u_k \rangle \ge \langle J_{k-1} u_k, u_k \rangle$ for any $k \ge 1$. Therefore, (24) implies that

$$\begin{aligned} f(x_{k+1}) - f(x_k) \le \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \langle G_k(x_{k+1} - x_k), x_{k+1} - x_k \rangle \\ + L_H(r_k + r_{k-1})\|x_{k+1} - x_k\|^2 - \frac{\mu}{2}\|x_{k+1} - x_k\|^2 \,. \end{aligned} \tag{25}$$

Since x_{k+1} is the solution of the inclusion (19) and g is convex, the subgradient inequality holds:

$$g(x_{k+1}) - g(x_k) \le \langle v_{k+1}, x_{k+1} - x_k \rangle \,. \tag{26}$$

Combining (25) and (26), we deduce that

$$\begin{aligned} F(x_{k+1}) - F(x_k) &\le \langle v_{k+1} + \nabla f(x_k), x_{k+1} - x_k \rangle + \langle G_k(x_{k+1} - x_k), x_{k+1} - x_k \rangle \\ &\quad + L_H(r_k + r_{k-1})\|x_{k+1} - x_k\|^2 - \frac{\mu}{2}\|x_{k+1} - x_k\|^2 \\ &= -\frac{\mu}{2} r_k^2 \,, \end{aligned} \tag{27}$$

where the last equality holds due to (19). $\square$

Due to the above descent lemma of function values, we can derive bounds for several sequences.

Lemma 4.7. *Given a number of iterations $N \in \mathbb{N}$, we have:*

$$\sum_{k=0}^{N} r_k^2 \le \frac{2(F(x_0) - \inf F)}{\mu} < +\infty\,, \tag{28}$$

$$\sum_{k=0}^{N-1} r_k \le C_0 N^{\frac{1}{2}}\,, \tag{29}$$

$$\sum_{k=1}^{N} r_k \le C_0 N^{\frac{1}{2}}\,, \tag{30}$$

where $C_0 := \sqrt{\frac{2(F(x_0)-\inf F)}{\mu}}$.

Proof. The first inequality is derived from Lemma 4.6. Then, we apply the Cauchy–Schwarz inequality and obtain

$$\sum_{i=0}^{N-1} r_k \le \left(\sum_{k=0}^{N-1} r_k^2\right)^{1/2} \left(\sum_{i=0}^{N-1} 1^{\frac{2}{1}}\right)^{1/2} \le \left(\sum_{k=0}^{\infty} r_k^2\right)^{1/2} N^{1/2} \le C_0 N^{1/2}\,. \tag{31}$$

Similarly, we have

$$\sum_{i=1}^{N} r_k \le \left(\sum_{k=1}^{N} r_k^2\right)^{1/2} \left(\sum_{i=1}^{N} 1^{\frac{2}{1}}\right)^{1/2} \le \left(\sum_{k=0}^{\infty} r_k^2\right)^{1/2} N^{1/2} \le C_0 N^{1/2}\,. \tag{32}$$

□

Now, we provide several simple but useful inequalities showing the relations among G_k, $\tilde{G}_k$, J_k. Based on the growth rate of the summation over sequence $(r_k)_{k\in\mathbb{N}}$, we can derive an upper bound growth rate of the sequence $(\lambda_k)_{k\in\mathbb{N}}$.

Lemma 4.8. *Given the number of iterations $N \in \mathbb{N}$, we have*

$$\sum_{k=1}^{N} \lambda_k \le 2L_H C_0 N^{1/2}$$

with C_0 as in Lemma 4.7.

Proof. The result follows by combining the definition of λ_k with Lemma 4.7.

□

Lemma 4.9. *For any $k \in \mathbb{N}$, we have*

$$u_k^\top \tilde{G}_k u_k \le \frac{1}{\mu}\|F'(x_k)\|^2\,. \tag{33}$$

Proof. Since g is convex, hence, its subdifferential monotone, we have $u_k^\top(v_{k+1}-v_k) \ge 0$. According to (19), we obtain that $\tilde{G}_k u_k = -\nabla f(x_k) - v_{k+1}$. Then we can deduce that:

$$\begin{aligned}
u_k^\top \tilde{G}_k u_k &\le u_k^\top \tilde{G}_k u_k + u_k^\top (v_{k+1} - v_k) \\
&= u_k^\top \left(-\nabla f(x_k) - v_{k+1} + v_{k+1} - v_k\right) \\
&= -u_k^\top F'(x_k) \\
&= -u_k^\top \tilde{G}_k^{1/2} \tilde{G}_k^{-1/2} F'(x_k) \\
&\le \|u_k\|_{\tilde{G}_k} \|F'(x_k)\|_{\tilde{G}_k^{-1}} \\
&\le \frac{1}{2} u_k^\top \tilde{G}_k u_k + \frac{1}{2}\|F'(x_k)\|^2_{\tilde{G}_k^{-1}} \\
&\le \frac{1}{2} u_k^\top \tilde{G}_k u_k + \frac{1}{2\mu}\|F'(x_k)\|^2\,,
\end{aligned} \tag{34}$$

where the second inequality uses Cauchy-Schwarz inequality, the third inequality uses Young's inequality and the last inequality holds since $\tilde{G}_k^{-1} \preceq \frac{1}{\mu} I$ (see Lemma 4.5 and Lemma 4.2). By rearranging the above inequality, we obtain the desired result. □

The following lemma is a crucial estimate of the descent of the corrected SR1 metric $\tilde{G}_k$ with respect to our potential function $V(\tilde{G}_k) = \operatorname{tr} \tilde{G}_k$.

Lemma 4.10. *For any $k \in \mathbb{N}$, we have the following descent inequality:*

$$V(\tilde{G}_k) - V(\tilde{G}_{k+1}) \geq \frac{\mu g_{k+1}^2}{g_k^2} - n\lambda_{k+1}\,, \tag{35}$$

where $g_k := \|F'(x_k)\|$.

Proof. By the optimality condition (19), we have $\tilde{G}_k u_k + v_{k+1} = -\nabla f(x_k)$ and by definition of J_k, we have $J_k u_k = \nabla f(x_{k+1}) - \nabla f(x_k)$. Using Lemma 4.4, we make the following estimation:

$$\begin{aligned} V(\tilde{G}_k) - V(G_{k+1}) = \nu(J_k, \tilde{G}_k, u_k) &\geq \frac{\|(\tilde{G}_k - J_k)u_k\|^2}{u_k^\top \tilde{G}_k u_k} = \frac{\|\nabla f(x_{k+1}) + v_{k+1}\|^2}{u_k^\top \tilde{G}_k u_k} \\ &= \frac{\|F'(x_{k+1})\|^2}{u_k^\top \tilde{G}_k u_k} \geq \frac{\mu \|F'(x_{k+1})\|^2}{\|F'(x_k)\|^2} = \frac{\mu g_{k+1}^2}{g_k^2}\,, \end{aligned} \tag{36}$$

where the second inequality holds due to Lemma 4.9. Summing this with

$$V(G_{k+1}) - V(\tilde{G}_{k+1}) = \operatorname{tr}(G_{k+1} - G_{k+1} - \lambda_{k+1} I) \geq -n\lambda_{k+1}\,, \tag{37}$$

we obtain the desired inequality.

□

Now, we are ready to prove our main theorem for Algorithm 1.

Proof of Theorem 3.1. For simplicity, we denote $\gamma_k^2 := \frac{g_{k+1}^2}{g_k^2}$, where $g_k := \|F'(x_k)\|$. By Lemma 4.10, summing (35) from $k = 0$ to N, we obtain

$$V(\tilde{G}_0) - V(\tilde{G}_N) \geq \mu \sum_{k=0}^{N-1} \gamma_k^2 - n \sum_{k=1}^{N} \lambda_k\,. \tag{38}$$

Since, for every k, our method preserves the relation $\tilde{G}_k \succeq G_{k+1} \succeq J_k$ (see Lemma 4.5), we have $V(\tilde{G}_k) > 0$ for all $k \in \mathbb{N}$. Therefore, we deduce that

$$V(\tilde{G}_0) \geq \mu \sum_{k=0}^{N-1} \gamma_k^2 - n \sum_{k=1}^{N} \lambda_k\,. \tag{39}$$

By Lemma 4.8, we obtain

$$CN^{1/2} \geq \frac{1}{\mu}\left(V(G_0) + 2nL_H C_0 N^{1/2}\right) \geq \sum_{k=0}^{N-1} \gamma_k^2\,. \tag{40}$$

where $C := (2nL_H C_0 + V(G_0))/\mu$. Dividing (40) by N on both sides, we deduce with the concavity and monotonicity of $\log x$ that

$$\log \frac{C}{N^{1/2}} \geq \log\left(\frac{1}{N}\sum_{k=0}^{N-1} \gamma_k^2\right) \geq \frac{1}{N}\sum_{k=0}^{N-1} \log(\gamma_k^2) \geq \log\left(\left(\prod_{k=0}^{N-1} \frac{g_{k+1}^2}{g_k^2}\right)^{1/N}\right) = \log\left(\left(\frac{g_N}{g_0}\right)^{2/N}\right)\,. \tag{41}$$

Thus, we obtain the result

$$g_N \leq \left(\frac{C}{N^{1/2}}\right)^{N/2} g_0\,. \tag{42}$$

□

4.3 Convergence analysis of Algorithm 2

Now, let us analyze Algorithm 2. The optimality condition of the update step in (5) is

$$\nabla f(x_k) + v_{k+1} + \tilde{G}_k(x_{k+1} - x_k) = 0\,, \tag{43}$$

where $v_{k+1} := -\nabla f(x_k) - \tilde{G}_k(x_{k+1} - x_k) \in \partial g(x_{k+1})$.

As in the previous section, we need several bounds for sequences $(\tilde{G}_k)_{k\in\mathbb{N}}$, $(G_k)_{k\in\mathbb{N}}$ and $(J_k)_{k\in\mathbb{N}}$.

Lemma 4.11. *For each $k \in \mathbb{N}$, we have*

$$\mu I \preceq J_k \preceq G_{k+1} \preceq \tilde{G}_k\,, \tag{44}$$

$$\operatorname{tr} G_{k+1} \leq \operatorname{tr} \tilde{G}_k \leq n\bar{\kappa}\,. \tag{45}$$

Proof. We first notice that since g is convex, the monotonicity of its subdifferential yields

$$u_k^\top (v_{k+1} - v_k) \geq 0\,,$$

for any $k \in \mathbb{N}$. Then, for any $k \in \mathbb{N}$, we have

$$u_k^\top \tilde{G}_k u_k \leq u_k^\top \tilde{G}_k u_k + u_k^\top (v_{k+1} - v_k) = u_k^\top (-\nabla f(x_k) - v_{k+1} + v_{k+1} - v_k) = -u_k^\top F'(x_k)\,. \tag{46}$$

The first inequality in (44) is a direct consequence of the strong convexity of f.

Now, we prove the remaining inequalities in (44) by induction. In order to validate the base case $k = 0$, we first observe that $\tilde{G}_0 = G_0(1 + \lambda_0) \succeq J_0$ holds by Lipschitz continuity of ∇f and $G_0 = LI$ and $\lambda_0 = 0$. Then, Lemma 4.3 shows the desired property:

$$J_0 \preceq G_1 = \mathrm{SR1}(J_0, \tilde{G}_0, u_0) \preceq \tilde{G}_0 \preceq nLI \preceq n\bar{\kappa}I\,.$$

For showing the induction, we suppose now that (44) holds for $k = N - 1$. We discuss by cases:

1. If $\operatorname{tr} \hat{G}_N \leq n\bar{\kappa}$, then, $\tilde{G}_N = \hat{G}_N$. Since $\tilde{G}_N = G_N(1 + \lambda_N) \geq \mu\lambda_N$ and Cauchy–Schwarz inequality, the above equality (46) implies

$$\mu\lambda_N \|u_N\|^2 \leq u_N^\top \tilde{G}_N u_N = -u_N^\top F'(x_N) \leq \|u_N\| \|F'(x_N)\|\,. \tag{47}$$

   Thus, $\|u_N\| \leq \frac{\|F'(x_N)\|}{\mu\lambda_N}$. Therefore,

$$L_H r_N = L_H \|u_N\| \leq \frac{L_H \|F'(x_N)\|}{\mu\lambda_N} \leq \sqrt{L_H \|F'(x_N)\|}\,, \tag{48}$$

   where the last inequality holds since $\mu\lambda_N \geq \sqrt{L_H\|F'(x_N)\|}$ (see the definition of λ_k). We can deduce that:

$$L_H r_N + L_H r_{N-1} \leq \sqrt{L_H \|F'(x_N)\|} + L_H r_{N-1} = \mu\lambda_N\,. \tag{49}$$

   Therefore, we deduce by the induction hypothesis and Lemma 4.2 that

$$\begin{aligned}\tilde{G}_N &= G_N(1 + \lambda_N)\\ &\succeq J_{N-1}(1 + \lambda_N)\\ &\succeq J_{N-1} + \mu\lambda_N I\\ &\succeq J_{N-1} + (L_H r_{N-1} + L_H r_N)I\\ &\succeq J_N\,.\end{aligned} \tag{50}$$

   Using Lemma 4.3 again, we obtain $J_N \preceq G_{N+1} = \mathrm{SR1}(J_N, \tilde{G}_N, u_N) \preceq \tilde{G}_N$ and $\operatorname{tr} G_{N+1} \leq \operatorname{tr} \tilde{G}_N \leq n\bar{\kappa}$.

2. If $\operatorname{tr} \hat{G}_N > n\bar{\kappa}$, then $\tilde{G}_N = LI$ due to the restarting step. Automatically, we have $\tilde{G}_N \succeq J_N$ and using Lemma 4.3 again, we obtain $J_N \preceq G_{N+1} \preceq \tilde{G}_N \preceq LI$ and $\operatorname{tr} G_{N+1} \leq \operatorname{tr} \tilde{G}_N = nL \leq n\bar{\kappa}$. □

Remark 4.12. *Lemma 4.11 indicates that both G_{k+1} and $\tilde{G}_k$ are bounded by $n\bar{\kappa}I$ for any $k \in \mathbb{N}$ due to the restarting strategy, while the candidate $\hat{G}_k$ has the possibility to be larger than $n\bar{\kappa}I$ where $\bar{\kappa} \geq L$. Since our potential function is simple and does not involve any Hessian term, the computational cost of the simple restarting criterion $\operatorname{tr} \hat{G}_{k+1} \leq n\bar{\kappa}$ is only $O(n)$, which is another advantage of our potential function.*

Based on the lemma above, we can derive a descent lemma for the objective values.

Lemma 4.13 (descent lemma for function values)**.** *For any $k \in \mathbb{N}$, we have*

$$F(x_{k+1}) - F(x_k) \leq -\frac{\mu}{2} r_k^2 \,. \tag{51}$$

Proof. The argument is similar with the one of Lemma 4.6. By the strong convexity of f, we have

$$\begin{aligned} f(x_{k+1}) - f(x_k) &\leq \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \langle \nabla f(x_{k+1}) - \nabla f(x_k), x_{k+1} - x_k \rangle - \frac{\mu}{2}\|x_{k+1} - x_k\|^2 \\ &= \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \langle J_k(x_{k+1} - x_k), x_{k+1} - x_k \rangle - \frac{\mu}{2}\|x_{k+1} - x_k\|^2 \\ &\leq \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \left\langle \tilde{G}_k(x_{k+1} - x_k), x_{k+1} - x_k \right\rangle - \frac{\mu}{2}\|x_{k+1} - x_k\|^2 \,, \end{aligned} \tag{52}$$

where the second inequality above holds due to Lemma 4.11. Since x_{k+1} is the solution of (43) and $g(x)$ is convex, we can derive that

$$g(x_{k+1}) - g(x_k) \leq \langle v_{k+1}, x_{k+1} - x_k \rangle \,. \tag{53}$$

Combining (52) and (53), we deduce that

$$\begin{aligned} F(x_{k+1}) - F(x_k) &\leq \langle v_{k+1} + \nabla f(x_k), x_{k+1} - x_k \rangle + \left\langle \tilde{G}_k(x_{k+1} - x_k), x_{k+1} - x_k \right\rangle - \frac{\mu}{2}\|x_{k+1} - x_k\|^2 \\ &= -\frac{\mu}{2} r_k^2 \,, \end{aligned} \tag{54}$$

where the last equality holds due to (43). ∎

Next, we study the growth rates of several important sequences.

Lemma 4.14. *Given a number of iteration $N \in \mathbb{N}$, we have:*

$$\sum_{k=0}^{N} r_k^2 \leq \frac{2(F(x_0) - \inf F)}{\mu} < \infty \,; \tag{55}$$

$$\sum_{k=0}^{N-1} r_k \leq C_0 N^{1/2} \quad \textit{and} \quad \sum_{k=1}^{N} r_k \leq C_0 N^{1/2} \,; \tag{56}$$

$$\sum_{k=1}^{N} \lambda_k \leq \Theta N^{3/4} \quad \textit{where } \Theta := \frac{1}{\mu}(L_H C_0 + \sqrt{L_H(L + n\bar{\kappa})C_0}) \,, \tag{57}$$

where $C_0 := \sqrt{\frac{2(F(x_0) - \inf F)}{\mu}}$.

Proof. We start with the first and the second results. By summing the inequality in Lemma 4.13 from $k = 0$ to $N - 1$, we have

$$\frac{\mu}{2} \sum_{k=0}^{N-1} r_k^2 \leq \sum_{k=0}^{N-1} F(x_k) - F(x_{k+1}) \leq F(x_0) - \inf F \,, \tag{58}$$

which yields the first result. Using Cauchy–Schwarz inequality, we obtain:

$$\sum_{k=0}^{N-1} r_k \leq \left(\sum_{k=0}^{N-1} r_k^2 \right)^{1/2} \left(\sum_{k=0}^{N-1} 1 \right)^{1/2} \leq C_0 N^{1/2} \,, \tag{59}$$

and because of the uniform bound above, we conclude that

$$\sum_{k=1}^{N} r_k \leq \left(\sum_{k=1}^{N} r_k^2 \right)^{1/2} \left(\sum_{k=1}^{N} 1 \right)^{1/2} \leq C_0 N^{1/2} \,. \tag{60}$$

Now, we prove the third result. From (43) we derive that:

$$
\begin{aligned}
L_H\|F'(x_k)\| &= L_H\|\nabla f(x_k) - \nabla f(x_{k-1}) - \tilde{G}_{k-1}(x_k - x_{k-1})\| \\
&\le L_H(\|\nabla f(x_k) - \nabla f(x_{k-1})\| + \|\tilde{G}_{k-1}(x_k - x_{k-1})\|) \\
&\le L_H(L + n\bar{\kappa})r_{k-1}\,,
\end{aligned}
\tag{61}
$$

where the last inequality holds since ∇f is L-Lipschitz continuous and $\tilde{G}_k$ is bounded uniformly for any $k \in \mathbb{N}$ by the restarting step in Algorithm 2. For convenience, we denote $a_k := \sqrt{L_H\|F'(x_k)\|}$ for all $k \in \mathbb{N}$. From (61) and (55), we get

$$
\sum_{k=1}^{N} a_k^2 \le L_H(L + n\bar{\kappa}) \sum_{k=0}^{N-1} r_k \le L_H(L + n\bar{\kappa})C_0 N^{1/2}\,. \tag{62}
$$

Then, by Cauchy–Schwarz inequality, we obtain

$$
\sum_{k=1}^{N} a_k \le \left(\sum_{k=1}^{N} a_k^2\right)^{1/2} \left(\sum_{k=1}^{N} 1\right)^{1/2} \le \sqrt{L_H(L+n\bar{\kappa})C_0}N^{1/4} * N^{1/2} \le \sqrt{L_H(L+n\bar{\kappa})C_0}N^{3/4}\,. \tag{63}
$$

Thus, we have the growth rate:

$$
\sum_{k=1}^{N} \lambda_k = \frac{1}{\mu}\left(\sum_{k=1}^{N} L_H r_{k-1} + \sum_{k=1}^{N} a_k\right) \le \frac{1}{\mu}\left(L_H C_0 N^{1/2} + \sqrt{L_H(L+n\bar{\kappa})C_0}N^{3/4}\right) \le \Theta N^{3/4}\,, \tag{64}
$$

where $\Theta = \frac{1}{\mu}\left(L_H C_0 + \sqrt{L_H(L+n\bar{\kappa})C_0}\right)$. □

Lemma 4.15. *For any $k \in \mathbb{N}$, we have*

$$
u_k^\top \tilde{G}_k u_k \le \frac{1}{\mu}\|F'(x_k)\|^2\,. \tag{65}
$$

Proof. According to (43), we obtain that $\tilde{G}_k u_k = -\nabla f(x_k) - v_{k+1}$. The rest of this proof remains the same as the one for Lemma 4.9. □

Lemma 4.16. *For any $k \in \mathbb{N}$, we have a descent inequality as the following:*

$$
V(\tilde{G}_k) - V(\tilde{G}_{k+1}) \ge \frac{\mu g_{k+1}^2}{g_k^2} - n\lambda_{k+1}\bar{\kappa}\,, \tag{66}
$$

where $g_k := \|F'(x_k)\|$.

Proof. By the optimality condition in (43), we have $\tilde{G}_k u_k + v_{k+1} = -\nabla f(x_k)$ and by definition of J_k, we have $J_k u_k = \nabla f(x_{k+1}) - \nabla f(x_k)$. Using Lemma 4.4, we have the following estimation:

$$
\begin{aligned}
V(\tilde{G}_k) - V(G_{k+1}) = \nu(J_k, \tilde{G}_k, u_k) &\ge \frac{\|(\tilde{G}_k - J_k)u_k\|^2}{u_k^\top \tilde{G}_k u_k} = \frac{\|\nabla f(x_{k+1}) + v_{k+1}\|^2}{u_k^\top \tilde{G}_k u_k} \\
&= \frac{\|F'(x_{k+1})\|^2}{u_k^\top \tilde{G}_k u_k} \ge \frac{\mu\|F'(x_{k+1})\|^2}{\|F'(x_k)\|^2} = \frac{\mu g_{k+1}^2}{g_k^2}\,,
\end{aligned}
\tag{67}
$$

where the second inequality holds due to Lemma 4.15.

We also need a lower bound for $V(G_{k+1}) - V(\tilde{G}_{k+1})$, for which we need to take the two cases of the restarting Step 4 into account.

1. When $\operatorname{tr}\hat{G}_{k+1} \le n\bar{\kappa}$, we have $\tilde{G}_{k+1} = \hat{G}_{k+1}$, $\operatorname{tr} G_{k+1} \le n\bar{\kappa}$ and

$$
V(G_{k+1}) - V(\tilde{G}_{k+1}) = \operatorname{tr}(G_{k+1} - G_{k+1} - \lambda_{k+1}G_{k+1}) \ge -n\lambda_{k+1}\bar{\kappa}\,. \tag{68}
$$

2. When $\operatorname{tr}\hat{G}_{k+1} > n\bar{\kappa}$, we have $V(\hat{G}_{k+1}) > n\bar{\kappa} \geq nL = \operatorname{tr} LI$. Since in this case we set $\tilde{G}_{k+1} = LI$, we have $V(\hat{G}_{k+1}) \geq V(\tilde{G}_{k+1})$. According to Lemma 4.11, we have $\operatorname{tr} G_{k+1} \leq n\bar{\kappa}$. Since $\lambda_k \geq 0$, we deduce that

$$\begin{aligned} V(G_{k+1}) - V(\tilde{G}_{k+1}) &= V(G_{k+1}) - V(\hat{G}_{k+1}) + V(\hat{G}_{k+1}) - V(\tilde{G}_{k+1}) \\ &\geq V(G_{k+1}) - V(\hat{G}_{k+1}) \\ &\geq \operatorname{tr}(G_{k+1} - G_{k+1} - \lambda_{k+1} G_{k+1}) \\ &\geq -\lambda_{k+1} \operatorname{tr}(G_{k+1}) \\ &\geq -n\lambda_{k+1}\bar{\kappa}\,. \end{aligned} \tag{69}$$

Therefore, adding (67) with either (68) or (69) yields the desired inequality. □

Now, we are ready to prove our main theorem for Algorithm 2.

Proof of Theorem 3.4. For symplicity, we denote $\gamma_k^2 \coloneqq \frac{g_{k+1}^2}{g_k^2}$. Summing the result in Lemma 4.16 from $k = 0$ to $N-1$, we obtain

$$V(\tilde{G}_0) - V(\tilde{G}_N) \geq \mu \sum_{k=0}^{N-1} \gamma_k^2 - n\bar{\kappa} \sum_{k=1}^{N} \lambda_k\,. \tag{70}$$

Since, for every k, our method keeps $\tilde{G}_k \succeq J_k$ and $G_{k+1} \succeq J_k$ (see Lemma 4.11), we have $V(\tilde{G}_k) > 0$ and therefore

$$V(\tilde{G}_0) \geq \mu \sum_{k=0}^{N-1} \gamma_k^2 - n\bar{\kappa} \sum_{k=1}^{N} \lambda_k\,. \tag{71}$$

By Lemma 4.14, we obtain

$$C_{\text{grad}} N^{3/4} \geq \frac{1}{\mu}\left(V(G_0) + n\bar{\kappa}\Theta N^{3/4}\right) \geq \sum_{k=0}^{N-1} \gamma_k^2\,, \tag{72}$$

where $C_{\text{grad}} \coloneqq (n\bar{\kappa}\Theta + V(G_0))/\mu$. Dividing (72) by N, we obtain

$$\frac{C_{\text{grad}}}{N^{1/4}} \geq \frac{1}{N} \sum_{k=0}^{N-1} \gamma_k^2\,. \tag{73}$$

We derive from (73) with the concavity and monotonicity of $\log x$ that

$$\log\left(\frac{C_{\text{grad}}}{N^{1/4}}\right) \geq \log\left(\frac{1}{N}\sum_{k=0}^{N-1}\gamma_k^2\right) \geq \frac{1}{N}\sum_{k=0}^{N-1}\log(\gamma_k^2) = \log\left(\left(\prod_{k=0}^{N-1}\frac{g_{k+1}^2}{g_k^2}\right)^{1/N}\right) = \log\left(\left(\frac{g_N}{g_0}\right)^{2/N}\right). \tag{74}$$

Thus, we obtain

$$g_N \leq \left(\frac{C_{\text{grad}}}{N^{1/4}}\right)^{N/2} g_0\,. \tag{75}$$

□

5 Experiments

In the following section, we consider two regression applications from machine learning to provide also numerical evidence about the superior performance of our algorithms and thereby validating the global non-asymptotic super-linear rates of convergence. While our algorithms can solve possibly non-smooth additive composite optimization problems, we restrict ourselves to smooth problems, for which our algorithms are new as well. This is mainly due to the fact that the efficient solution of sub-problems in the non-smooth case needs additional careful investigations, possibly along the lines of [2], which we will tackle in future work.

5.1 Regularized log-sum-exp problem

For experiment, we test our methods: Cubic SR1 PQN (Algorithm 1) and Grad SR1 PQN (Algorithm 2) on the log-sum-exp problem with regularization:

$$\min_{x \in \mathbb{R}^n} f(x)\,, \quad f(x) = \log\left(\sum_{i=1}^{m} \exp\left(\langle a_i, x\rangle - b_i\right)\right) + \frac{\mu}{2}\|x\|^2\,, \tag{76}$$

where $a_1, a_2, \cdots, a_m \in \mathbb{R}^n$, $b_1, b_2, \cdots, b_m \in \mathbb{R}$ are given. The strong convexity modulus is $\mu = 1$. The Lipschitz constant is $L = \mu + 2\sum_{i=1}^{m}\|a_i\|^2$ and $L_H = 2$. In our experiment, $(a_i)_{i=1,\dots,m} \subset \mathbb{R}^n$ and $(b_i)_{i=1,\dots,m} \subset \mathbb{R}$ are generated randomly with $m = 500$ and $n = 200$. We set $\bar{\kappa} = L$ for Grad SR1 PQN. As depicted in Figure 1, while the starting point is far from the optimal solution, our methods achieve a super-linear rate of convergence. However, since cubic regularization is hard to compute, Cubic Newton [34] and Cubic SR1 PQN are not competitive when it comes to measuring the actual computation time. This is in contrast to the Heavy ball Method (HBF) [36] (with optimal damping rate $\beta = \frac{\sqrt{L}-\sqrt{\mu}}{\sqrt{L}+\sqrt{\mu}}$ and step size $\tau = \frac{4}{(\sqrt{L}+\sqrt{\mu})^2}$) and Gradient Descent (GD), which both have cheap cost per iteration. Our gradient regularized quasi-Newton method outperforms all other methods significantly, especially, in initial iterations. However, the estimation of convergence rates of our method is not sharp, despite our methods perform much better in practice, which is the price to pay for globality and non-smoothness.

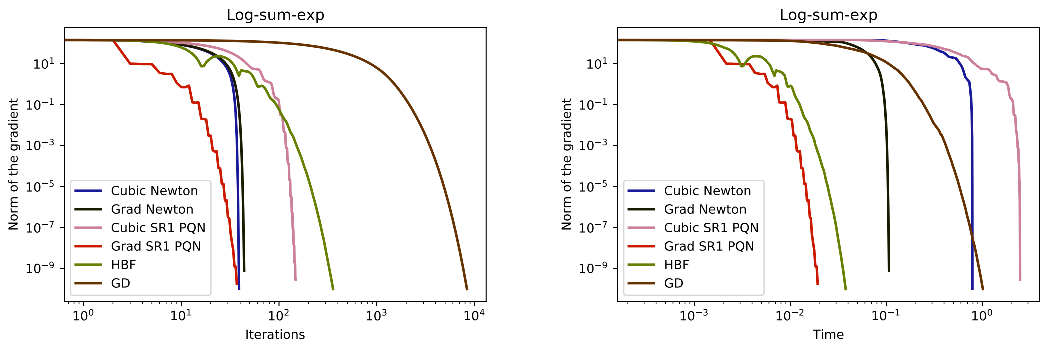

Figure 1: Our proposed Cubic SR1 PQN and Grad SR1 PQN significantly outperform first-order methods and even faster than related Newton methods, in the early iterations. In terms of time, our Grad SR1 PQN remains the fastest, while methods with cubic regularization take longer time when the initial point far from the optimal point.

5.2 Logistic regression

The second experiment is a logistic regression problem with regularization on the benchmark dataset "mushroom" from UCI Machine Learning Repository [42]:

$$\min_{x \in \mathbb{R}^n} f(x), \quad \text{with } f(x) := \frac{1}{m}\sum_{i=1}^{m} \log(1 + \exp(-b_i a_i^\top x)) + \frac{\mu}{2}\|x\|^2\,, \tag{77}$$

where $a_i \in \mathbb{R}^n$, $b_i \in \mathbb{R}$ for $i = 1, 2, \cdots, m$ denotes the given data from the "mushrooms" dataset. The strong convexity modulus is $\mu = 1$. The Lipschitz constants are $L = \mu + 2\sum_{i=1}^{m}\|a_i\|^2$ and $L_H = 2$. Here, $m = 8124$ and $n = 117$. We set $\bar{\kappa} = L$ for Grad SR1 PQN.

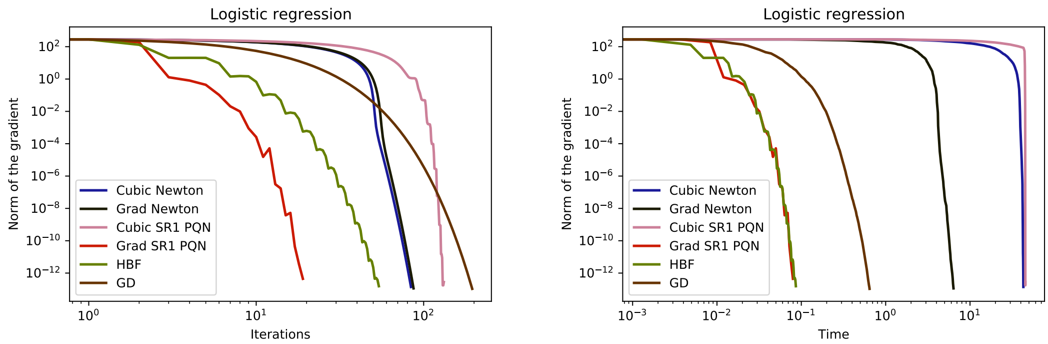

Figure 2: Our proposed Cubic SR1 PQN and Grad SR1 PQN still significantly outperform first-order methods, eventually, in number of iterations. In terms of time, our Grad SR1 PQN is still among the fastest methods due to SR1 metric and the cheap computation of the correction step and the restarting step.

As depicted in Figure 2, despite the starting point is far from the optimal solution, our methods achieve a super-linear rate of convergence globally. Since the computation for the quasi-Newton metric is less than that of Newton's method, Cubic quasi-Newton is better than Cubic Newton eventually in terms of time. However, due to the cubic term, neither Cubic Newton nor Cubic quasi-Newton is competitive when it comes to measuring the actual computation time. Our gradient regularized SR1 quasi-Newton method outperforms all other methods significantly in number of iterations and is among the fastest in terms of time.

6 Conclusion

In this paper, we propose two variants of proximal quasi-Newton SR1 methods which converge globally with an explicit (non-asymptotic) super-linear rate of convergence. The key is the adaptation of cubic and gradient regularization from the pioneering works [34, 33] on Newton's method to quasi-Newton methods. The potential function in our paper differs from previous works [40, 25, 44] and turns out to be more tractable for the analysis of the regularized SR1 quasi-Newton method, also by making the restarting step numerically easier. Moreover, our algorithm and analysis is directly developed for non-smooth additive composite problems. The analysis of the convergence used in this paper paves the road to generalize many regularized Newton methods, in particular, [33, 17, 16]. Since this paper considers the SR1 method, an interesting generalization is that of proximal quasi-Newton methods with BFGS or DFP metric update under the same conditions, especially without using line search.

A A variant of gradient regularized proximal quasi-Newton method

Algorithm 3 is almost the same with Algorithm 2. Step 1 of Algorithm 3 equips the classical quasi-Newton term with an additive gradient regularization term instead of a scaling factor involving gradient. Step 2 defines J_k as in Algorithm 2. In Step 3, several variables are defined, including $F'(x_{k+1})$ which will be used to generate the regularization term, compute λ_k and terminate the algorithm in Step 5. Step 4 computes the corrected metric $\hat{G}_{k+1}$ by adding λ_k. As in Algorithm 2, we adopt the same restarting strategy for $\tilde{G}_{k+1}$ in Algorithm 3.

Algorithm 3 Grad Reg SR1 PQN

Require: x_0, $G_0 = LI$, $\lambda_0 = 0$, $\tilde{G}_0 = G_0 + \lambda_0$, $\bar{\kappa} \geq L$.

Update for $k = 0, \cdots, N$:

1. **Update**

$$x_{k+1} = \underset{x \in \mathbb{R}^n}{\operatorname{argmin}}\, g(x) + \langle \nabla f(x_k), x - x_k \rangle + \frac{1}{2}\|x - x_k\|^2_{\tilde{G}_k} . \tag{78}$$

2. **Denote but not use explicitly** $J_k \coloneqq \int_0^1 \nabla^2 f(x_k + tu_k)dt$.
3. **Compute** $u_k = x_{k+1} - x_k$, $r_k = \|u_k\|$ and

$$\begin{aligned} F'(x_{k+1}) &= \nabla f(x_{k+1}) - \nabla f(x_k) - \tilde{G}_k u_k \\ G_{k+1} &= \mathrm{SR1}(J_k, \tilde{G}_k, u_k) \\ \lambda_{k+1} &= \left(\sqrt{L_H \|F'(x_{k+1})\|} + L_H r_k \right) . \end{aligned}$$

4. **Update** $\tilde{G}_{k+1}$**:** compute $\hat{G}_{k+1} = G_{k+1} + \lambda_{k+1}$ (**Correction step**).

   If $\operatorname{tr} \hat{G}_{k+1} \leq n\bar{\kappa}$**:** we set $\tilde{G}_{k+1} = \hat{G}_{k+1}$.

   Othewise: we set $\tilde{G}_{k+1} = LI$ (**Restart step**).
5. **If** $\|F'(x_{k+1})\| = 0$: Terminate.

End

In Algorithm 3, we have $\tilde{G}_k = G_k + \lambda_k$ where G_k is generated by the SR1 method or $\tilde{G}_k = LI$. We consider the smooth case that $g = 0$. When $\tilde{G}_k = G_k + \lambda_k$, we can not use Sherman-Morrison-Woodbury formula to derive the inverse of $\tilde{G}_k$ due to the addition. Thus, even for smooth case, the computational cost of update step is the same as the inversion of a matrix, namely, $O(n^3)$. However, for non-smooth setting, the computational cost of the update step in Algorithm 3 is the same as that of classical proximal Newton method. For Algorithm 3, we have the following non-asymptotic super-linear rate.

Theorem A.1 (Main Theorem 3)**.** *For any initialization $x_0 \in \mathbb{R}^n$ and any $N \in \mathbb{N}$, Grad Reg SR1 PQN has a global convergence with the rate:*

$$\|F'(x_N)\| \leq \left(\frac{C_{\mathrm{grad}}}{N^{1/4}} \right)^{N/2} \|F'(x_0)\| , \tag{79}$$

where $C_{\mathrm{grad}} \coloneqq (\Theta + nL)/\mu$, $C_0 \coloneqq \sqrt{\frac{2(F(x_0) - \inf F)}{\mu}}$ and $\Theta = L_H C_0 + \sqrt{L_H (L + n\bar{\kappa}) C_0}$.

Proof. See Section B. □

Remark A.2. *In fact, (79) indicates that the super-linear rate is only attained after $N \geq C^4_{\mathrm{grad}}$ number of iterations, where $C_{\mathrm{grad}} = O(\frac{n}{\mu^{3/2}})$.*

B Convergence analysis of Algorithm 3

Let $\lambda_k \coloneqq \left(L_H r_{k-1} + \sqrt{L_H \|F'(x_k)\|} \right)$. We obtain a gradient regularized SR1 proximal quasi-Newton method which can be also regarded as a generalization of gradient regularized proximal Newton method [17]. The convergence analysis is very similar with the one for Algorithm 2 since the only difference in Algorithm 3 is that $\tilde{G}_{k+1} = G_{k+1} + \lambda_{k+1}$ with $\lambda_{k+1} = \left(2L_H r_{k-1} + \sqrt{2L_H \|F'(x_k)\|} \right)$. In this case, we have $\sum_{k=1}^N \lambda_k \leq \Theta N^{3/4}$ where $\Theta = L_H C_0 + \sqrt{L_H (L + n\bar{\kappa}) C_0}$ is a constant. The proof of that property is very similar with the one of Lemma 4.14. Most properties for Algorithm 2 listed in previous section still hold true for Algorithm 3 except the following lemma.

Lemma B.1. *For any $k \in \mathbb{N}$, we have a descent inequality as the following:*

$$V(\tilde{G}_k) - V(\tilde{G}_{k+1}) \geq \frac{\mu g^2_{k+1}}{g^2_k} - n\lambda_{k+1} , \tag{80}$$

where $g_k := \|F'(x_k)\|$.

Proof. The only difference from the proof of Lemma 4.16 lies in the estimation of $V(G_{k+1}) - V(\tilde{G}_{k+1})$. However, we have to discuss by cases due to the restart step.

1. When $\operatorname{tr}\hat{G}_{k+1} \leq n\bar{\kappa}$, we have $\tilde{G}_{k+1} = \hat{G}_{k+1}$, $\operatorname{tr}G_{k+1} \leq n\bar{\kappa}$ and

$$V(G_{k+1}) - V(\tilde{G}_{k+1}) = \operatorname{tr}(G_{k+1} - G_{k+1} - \lambda_{k+1}I) \geq -n\lambda_{k+1}\,. \tag{81}$$

2. When $\operatorname{tr}\hat{G}_{k+1} > n\bar{\kappa}$, we have $V(\hat{G}_{k+1}) > n\bar{\kappa} \geq nL$. Since in this case we set $\tilde{G}_{k+1} = LI$, we have $V(\hat{G}_{k+1}) \geq V(\tilde{G}_{k+1})$. We deduce that

$$\begin{aligned} V(G_{k+1}) - V(\tilde{G}_{k+1}) &= V(G_{k+1}) - V(\hat{G}_{k+1}) + V(\hat{G}_{k+1}) - V(\tilde{G}_{k+1}) \\ &\geq V(G_{k+1}) - V(\hat{G}_{k+1}) \\ &\geq \operatorname{tr}(G_{k+1} - G_{k+1} - \lambda_{k+1}I) \\ &\geq -n\lambda_{k+1} \\ &\geq -n\lambda_{k+1}\,. \end{aligned} \tag{82}$$

Therefore, no matter which case, we always have

$$V(G_{k+1}) - V(\tilde{G}_{k+1}) \geq -n\lambda_{k+1}\,. \tag{83}$$

Summing (83) and (67), we obtain the desired inequality. □

Now, we are ready to prove our main theorem for Algorithm 3.

Proof of Theorem A.1. Following the same steps of the proof of Theorem 3.4 and using Lemma B.1, we obtain the global superlinear convergence rate (Theorem A.1) of Algorithm 3. □

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