

# Convergence rates of regularized quasi-Newton methods without strong convexity

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August 20, 2025

## Abstract

In this paper, we study convergence rates of the cubic regularized proximal quasi-Newton method (Cubic SR1 PQN) for solving non-smooth additive composite problems that satisfy the so-called Kurdyka-Łojasiewicz (KL) property with respect to some desingularizing function  $\varphi$  rather than strong convexity. After a number of iterations  $k_0$ , Cubic SR1 PQN exhibits non-asymptotic explicit super-linear convergence rates. In particular, when  $\varphi(t) = ct^{1/2}$  with  $c$  being a positive constant, Cubic SR1 PQN has a convergence rate of order  $\left(\frac{C}{(k-k_0)^{1/2}}\right)^{(k-k_0)/2}$ , where  $k$  is the number of iterations and  $C > 0$  is a constant. For functions satisfying the global Łojasiewicz inequality, the rate becomes global (and non-asymptotic). To the best of our knowledge, this work presents, for the first time, such convergence rates of regularized (proximal) SR1 methods for minimizing non-convex non-smooth objectives satisfying the KL property. Actually, the rates are novel even in the smooth non-convex case. Notably, we achieve this without employing line search or trust region strategies, without assuming the Dennis–Moré condition, without any assumptions on quasi-Newton metrics and without assuming strong convexity. Moreover, for convex problems, we focus on a more tractable gradient regularized quasi-Newton method (Gradient SR1 PQN) which can achieve results similar to those obtained with cubic regularization. We also demonstrate, for the first time, non-asymptotic super-linear convergence rate of Gradient SR1 PQN for solving convex problems with the help of the Łojasiewicz inequality instead of strong convexity.

## 1 Introduction

Quasi-Newton methods have been studied extensively over decades due to their fast convergence. They have been developed to solve smooth optimization problems by only approximating the Hessian information (second derivative) or its inverse of the objective [21], while solely relying on first-order information. Based on this idea, numerous variants have been developed, including BFGS [12], SR1 [19, 11], DFP [19, 25]. A striking advantage of quasi-Newton-type methods [12, 13, 22, 24, 27, 15, 14] for minimizing some strongly convex smooth function  $f$ , as compared to generic first order schemes, is their local super-linear convergence. However, the convergence rates obtained by all works cited above are asymptotic, i.e.,  $\lim_{k \rightarrow \infty} \frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|} \rightarrow 0$  or  $\lim_{k \rightarrow \infty} \frac{\|\nabla f(x_{k+1})\|}{\|\nabla f(x_k)\|} \rightarrow 0$ . Since an asymptotic statement is about the limit as  $k \rightarrow \infty$ , we are also interested in a non-asymptotical analysis where the convergence rate is valid for any  $k$  or at least for all  $k \geq k_0$  where  $k_0$  is some constant. Only a non-asymptotic statement can characterize an explicit upper bound on the error of quasi-Newton methods. Non-asymptotic explicit local super-linear convergence rates for classical quasi-Newton schemes, including BFGS and SR1, applied to strongly convex objectives, were established only recently in [49, 50, 31, 56]. The guarantee is, however, only local, meaning that the initial point must be close to the minimum point. To obtain global rates, [48, 30] adopt BFGS with line-search strategies, while [54] studies the SR1 method on a composite problem with a regularization strategy.

In this paper, we would like to generalize the results from [54] to problems without strong convexity. A possible way is to assume that the objective function  $f$  has the Kurdyka-Łojasiewicz

(KL) property [6, 8, 38, 45]. When  $f$  verifies the global Łojasiewicz inequality, convergence rates of first-order methods were obtained in [9, 1, 26] both in the convex and non convex case. These are functions verifying a gradient domination inequality [42].

In this paper, we derive non-asymptotic convergence rates by studying cubic- and gradient-regularized quasi-Newton methods from [54] with the help of the KL property. This paper consists of two parts. In the first part, we provide convergence analysis of cubic regularized PQN on non-convex non-smooth problems under a general desingularization function  $\varphi$  or  $\varphi = ct^{1-\theta}$  for  $\theta \in (0, 1)$ . In particular, when  $\theta \in (0, \frac{1}{2})$  in the nonconvex case, we show an explicit super-linear convergence rate  $O\left(\frac{C}{k^{1/2}}\right)^{k/2}$  for some  $C > 0$  and  $k$  large enough. In the second part, we turn to the convex case and use more tractable gradient regularization, rather than cubic regularization, to achieve similar results. The main contribution of this paper is to reveal that KL inequality is the key property for the fast convergence of quasi-Newton methods. Though Dennis–Moré condition is widely assumed [17, 52, 15] to show superlinear convergence, our convergence analysis does not rely on this usually hard-to-verify condition. Furthermore, we have no assumptions on our quasi-Newton metrics either. We also do not adopt any line search or trust region strategies to achieve global convergence.

## 2 Related Works

**Regularized Newton method.** Although global convergence can be obtained in the convex case while maintaining local fast convergence [20, 43] by using line search or trust region strategies, there are regularization strategies to achieve the same goal by adding a positive definite matrix to the Hessian (like in the Levenberg–Marquardt method [37, 40]). Thanks to positive definiteness of the metric, Newton-type methods are stabilized at the cost of loosing the local fast convergence rates. The pioneering work by Nesterov and Polyak [42] has introduced cubic regularization as another globalization strategy that avoids line search and at the same time shows the standard fast local convergence rates of the pure Newton’s method. This cubic regularization works even on non-convex problems such as star-convex functions or gradient dominated functions, namely, those satisfying the condition that for any  $x \in \text{dom } f$ ,  $f(x) - \min f \leq C\|\nabla f(x)\|^p$  with some  $C > 0, p > 0$ . Building on this work, Cartis et al. [15, 16] proposed an adaptive cubic regularization approach that retains comparable results while assuming only local Lipschitz continuity of the Hessian. In order to remedy the computational cost of cubic regularization, in [41, 23] a gradient regularization strategy was introduced for solving convex problems. It also allows for global convergence with a super-linear rate of convergence for strongly convex and smooth problems. Motivated by their works, [54] proposed two regularized SR1 proximal quasi-Newton methods with non-asymptotic super-linear convergence rate on strongly convex problems *We will study the non-asymptotic convergence rates of two algorithms from [54] without the need of strong convexity although it played a crucial role in the proofs of [54].*

**Quasi-Newton methods.** Rodomanov and Nesterov [49] obtained the first local explicit super-linear convergence rate for greedy quasi-Newton methods. Later, Ye et al. [56] obtained the first local explicit super-linear convergence rate for the SR1 quasi-Newton method. Building on this, [54] studied global rates. Both works significantly inspired our development in this paper. However, all results above require strong convexity of the objective function. *In our paper, we study (cubic, gradient) regularized proximal SR1 methods from [54], showing non-asymptotic super-linear rates without strong convexity, or even in the no-convex setting.*

There are a few more works, appearing under similar names to ‘regularized Newton-type method’ [33, 4, 5, 39, 28]. Since they do not consider non-asymptotic convergence rates and are using line search or trust region strategies, we do not describe them in more detail here. As for cubic regularization, in the well-known work [15], the authors also considered asymptotic convergence of cubic regularized quasi-Newton method on non-convex smooth problems. They assumed the Dennis–Moré condition that played a crucial rule in their proofs. There is another way to develop cubic regularized inexact Newton method without the explicit computation of Hessian via using finite differences [29]. Quasi-Newton methods with cubic regularization were also studied in [32] for star-convex functions. However, super-linear convergence was not investigated there.

**Proximal quasi-Newton and Newton-type methods.** One of the earliest work on proximal quasi-Newton (PQN) methods is [17], where they adopted line search to guarantee global convergence and assume the Dennis–Moré criterion to obtain asymptotic super-linear convergence. [36] also demonstrated an asymptotic super-linear local convergence rate of the proximal Newton and a proximal quasi-Newton method under the same condition. Both papers were dedicated to the convex case. We believe that the Dennis–Moré criterion is very restrictive and hard to check. Later, [51] showed that by a prox-parameter update mechanism, rather than a line search, they can derive a sub-linear global complexity on convex problems. However, to the best of our knowledge, there is no result on non-asymptotic explicit super-linear convergence rates for a proximal quasi-Newton method on composite problems without strong convexity. While we focus on the convergence rate, the efficient evaluation of the proximal mapping with respect to the variable metric is also crucial. There are a few works [51, 35, 2] that are associated to this topic. In particular, [2] proposed a proximal calculus to tackle this difficulty for the SR1 metric, followed by [3, 55, 33].

### 3 Preliminaries

#### 3.1 Notation

We denote  $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$  and  $\mathbb{R}_+ = [0, \infty)$ .  $\mathbb{R}^n$  is equipped with the standard inner product  $\langle u, v \rangle = u^\top v$  and the associated norm  $\|u\| := \sqrt{u^\top u}$ , for any  $u$  and  $v \in \mathbb{R}^n$ . Given symmetric positive definite  $M \in \mathbb{R}^{n \times n}$ , we denote  $\|u\|_M = \sqrt{u^\top M u}$ . Given a matrix  $A \in \mathbb{R}^{n \times n}$ ,  $\|A\|$  denotes the matrix norm induced by the Euclidean vector norm. We write  $\text{tr } A$  for the trace of the matrix  $A \in \mathbb{R}^{n \times n}$ . Moreover,  $I$  denotes the identity matrix in  $\mathbb{R}^n$ . We adopt the standard definition of the Loewner partial order of symmetric positive semi-definite matrices:  $A$  and  $B$  be two symmetric positive semi-definite matrices, we say  $A \preceq B$  (or  $A \prec B$ ) if and only if for any  $u \in \mathbb{R}^n$ , we have  $u^\top (B - A)u \geq 0$  (or  $u^\top (B - A)u > 0$ , respectively). Let  $F: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$  be a function. The domain of  $F$  is defined as  $\text{dom } F = \{x | F(x) < +\infty\}$ . For any  $\alpha, \beta \in (0, \infty]$  with  $\alpha < \beta$ , we denote the set  $\{x | \alpha < F(x) < \beta\}$  by  $[\alpha < F < \beta]$ . Let  $(x_k)_{k \in \mathbb{N}}$  be a sequence in  $\mathbb{R}^n$  starting from  $x_0$ . The set of all clusters of  $(x_k)_{k \in \mathbb{N}}$  is denoted by  $\omega(x_0)$ , i.e.,

$$\omega(x_0) = \{x \in \mathbb{R}^n | \exists (x_{k_q})_{q \in \mathbb{N}} \text{ such that } x_{k_q} \rightarrow x \text{ as } q \rightarrow \infty\}.$$

We say a function  $F: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$  is lower semicontinuous at  $\bar{x}$  if

$$\liminf_{x \rightarrow \bar{x}} F(x) \geq F(\bar{x}),$$

and lower semicontinuous (lsc) on  $\mathbb{R}^n$  if this holds for every  $\bar{x} \in \mathbb{R}^n$ .

#### 3.2 Subdifferentials of non-convex and non-smooth functions

**Definition 3.1** (Subdifferentials [47]). Let  $F: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$  be a proper and lsc function.

1. For a given  $x \in \text{dom } F$ , the Fréchet subdifferential of  $F$  at  $x$ , written  $\hat{\partial}F(x)$ , is the set of all vectors  $u \in \mathbb{R}^n$  which satisfy

$$\liminf_{y \neq x, y \rightarrow x} \frac{F(y) - F(x) - \langle u, y - x \rangle}{\|y - x\|} \geq 0.$$

For  $x \notin \text{dom } F$ , we set  $\hat{\partial}F(x) = \emptyset$ .

2. The limiting-subdifferential or simply the subdifferential, of  $F$  at  $x \in \text{dom } F$ , written  $\partial F(x)$ , is defined through the following closure process

$$\partial F(x) := \{u \in \mathbb{R}^n : \exists x_k \rightarrow x, F(x_k) \rightarrow F(x) \text{ and } u_k \rightarrow u \text{ where } u_k \in \hat{\partial}F(x_k) \text{ as } k \rightarrow \infty\}.$$

We denote  $\text{crit}F := \{x | \partial F(x) \ni 0\}$ . We now recall the non-smooth Kurdyka–Łojasiewicz (KL) inequality introduced in [6]. Let  $\eta \in (0, +\infty]$ . We denote by  $\Phi_\eta$  the class of concave functions  $\varphi: [0, \eta] \rightarrow \mathbb{R}_+$  which satisfy the following conditions:  $\varphi: [0, \eta] \rightarrow \mathbb{R}_+$  is continuous;  $\varphi(0) = 0$ ;  $\varphi$  is continuously differentiable on  $(0, \eta)$  with  $\varphi'(s) > 0$ , for any  $s \in (0, \eta)$ .

**Definition 3.2** (Kurdyka-Łojasiewicz). A proper, lower semi-continuous function  $F$  is said to have the KL property locally at  $\bar{x} \in \text{dom} \partial F$  if there exist  $\eta \in (0, +\infty]$ , a neighborhood  $U$  of  $\bar{x}$  and a continuous concave function  $\varphi \in \Phi_\eta$  such that the Kurdyka-Łojasiewicz inequality holds, i.e.,

$$\varphi'(F(x) - F(\bar{x})) \text{dist}(0, \partial F(x)) \geq 1, \quad (1)$$

for all  $x$  in  $U \cap [F(\bar{x}) < F < F(\bar{x}) + \eta]$ . We say  $\varphi$  is a desingularizing function for  $F$  at  $\bar{x}$ . If  $F$  satisfies the KL inequality at any  $\bar{x} \in \text{dom} \partial F$ , then  $F$  is called a KL function.

All tame (definable) functions are KL functions [7]. In particular, for any  $\bar{x}$ , semi-algebraic functions have the KL property at any  $\bar{x}$  with  $\varphi$  of the form  $\varphi(t) = ct^{1-\theta}$  for some  $c \in \mathbb{R}_+$  and  $\theta \in (0, 1]$  [6]. Computing the KL exponent  $\theta$  is hard but crucial. We refer the readers to [38] for the calculus of KL exponent. Though the KL inequality is defined locally at each point, it can be uniformized thanks to a result of [10].

**Lemma 3.1** (Uniformized KL property [10]). *Let  $\Omega$  be a compact set and let  $F: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$  be a proper and lower semi-continuous function. Assume  $F$  is constant on  $\Omega$  and satisfies the KL property at each point of  $\Omega$ . Then, there exists  $\epsilon > 0$ ,  $\eta > 0$  and  $\varphi \in \Phi_\eta$  such that for all  $\bar{x} \in \Omega$  and all  $x$  in the set*

$$\{x \in \mathbb{R}^n \mid \text{dist}(x, \Omega) < \epsilon\} \cap [F(\bar{x}) < F < F(\bar{x}) + \eta], \quad (2)$$

one has

$$\varphi'(F(x) - F(\bar{x})) \text{dist}(0, \partial F(x)) \geq 1. \quad (3)$$

## 4 Problem Setup and Main Results

In this section, we describe the optimization problem we want to tackle with all standing assumptions and our main results. We postpone the technical parts of the convergence analysis to Section 5.

### 4.1 Problem Setup

In the whole paper, we consider the optimization problem:

$$\min_{x \in \mathbb{R}^n} F(x), \quad \text{where } F(x) := g(x) + f(x), \quad (4)$$

where we make the following assumptions:

**Assumption 1.** 1.  $F = g + f$  is bounded from below;

2.  $g: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$  is proper, lsc;

3.  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  is twice differentiable with  $L$ -Lipschitz gradient and  $L_H$ -Lipschitz Hessian.

4.  $\text{argmin } F \neq \emptyset$ .

**Assumption 2.**  $F$  is a KL function.

**Remark 4.1.** In our assumptions, neither  $f$  nor  $g$  is required to be convex.

### 4.2 Algorithms and main results

Let us first recap the classical symmetric rank-1 quasi-Newton (SR1) update  $G_+ := \text{SR1}(A, G, u)$ . Given two  $n \times n$  symmetric matrices  $A$  and  $G$  such that  $A \preceq G$  and  $u \in \mathbb{R}^n$ , we define the SR1 update as follows:

$$\text{SR1}(A, G, u) := \begin{cases} G, & \text{if } (G - A)u = 0; \\ G - \frac{(G-A)uu^\top(G-A)}{u^\top(G-A)u}, & \text{otherwise.} \end{cases} \quad (5)$$

### 4.2.1 General objectives

In order to solve the problem in (4), we study a cubic regularized proximal SR1 quasi-Newton method (Algorithm 1). Algorithm 1 (Cubic SR1 PQN) is similar to the first algorithm proposed in [54]. The only but important difference is that we will apply the same algorithm with a restarting strategy for non-convex problems. Step 1 has two cases. If  $\text{tr } G_k \leq n\bar{\kappa}$ , Step 1a is executed. Otherwise, Step 1b implements a special update step with a fixed metric  $LI$ . Step 2 defines  $J_k$  which is not used explicitly. Step 4 updates the quasi-Newton metric via SR1 method.

We obtain global convergence with the following local explicit rates.

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#### Algorithm 1 Cubic SR1 PQN

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**Require:**  $x_0, G_0 = LI, r_{-1} = 0, \bar{\kappa} \geq L$ .

**Update for**  $k = 0, \dots, N$ :

1. a. **If**  $\text{tr } G_k \leq n\bar{\kappa}$ : **update**

$$x_{k+1} \in \underset{x \in \mathbb{R}^n}{\text{argmin}} g(x) + \langle \nabla f(x_k), x - x_k \rangle + \frac{1}{2} \|x - x_k\|_{G_k + L_H r_{k-1} I}^2 + \frac{L_H}{3} \|x - x_k\|^3. \quad (6)$$

**Compute**  $u_k = x_{k+1} - x_k, r_k = \|u_k\|, \lambda_k = L_H(r_{k-1} + r_k)$ .

**Compute**  $\tilde{G}_{k+1} = G_k + \lambda_k I$  (**Correction step**).

b. **Otherwise: compute**

$$x_{k+1} \in \underset{x \in \mathbb{R}^n}{\text{argmin}} g(x) + \langle \nabla f(x_k), x - x_k \rangle + \frac{L + L_H r_{k-1}}{2} \|x - x_k\|^2 + \frac{L_H}{3} \|x - x_k\|^3. \quad (7)$$

**Compute**  $u_k = x_{k+1} - x_k, r_k = \|u_k\|, \lambda_k = L_H(r_{k-1} + r_k)$ .

**Compute**  $\tilde{G}_{k+1} = (L + \lambda_k)I$  (**Restart step**).

2. **Denote but do not use explicitly**  $J_k := \int_0^1 \nabla^2 f(x_k + t u_k) dt$ .

3. **Compute**

$$F'(x_{k+1}) = \nabla f(x_{k+1}) - \nabla f(x_k) - \tilde{G}_{k+1} u_k.$$

4. **Update quasi-Newton metric:**

$$G_{k+1} = \text{SR1}(J_k, \tilde{G}_{k+1}, u_k).$$

5. **If**  $\|F'(x_{k+1})\| = 0$ : **Terminate**.

**End**

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**Theorem 4.2.** *Let Assumption 1 and 2 hold. Assume the sequence  $(x_k)_{k \in \mathbb{N}}$  generated by Cubic SR1 PQN (Algorithm 1) is bounded. Then, we have  $\|F'(x_k)\| \rightarrow 0, F(x_k) \rightarrow \bar{F}$  for some  $\bar{F} \in \mathbb{R}$  and  $\text{dist}(x_k, \text{crit } F) \rightarrow 0$  as  $k \rightarrow \infty$ . Moreover, there exists  $k_0 \in \mathbb{N}$  such that for any  $k \geq k_0$ , Cubic SR1 PQN (Algorithm 1) has a local convergence rate:*

$$\left( \min_{\{i \in \mathbb{N} | k_0 \leq i \leq N + k_0\}} \|F'(x_i)\| \right) \leq \left( C_1 \left( \frac{C_{\text{CR1}}}{N} + \frac{C_{\text{CR2}}}{N^{1/2}} \right) \right)^{N/(N+1)} \|F'(x_{k_0})\|^{1/(N+1)}; \quad (8)$$

where  $N = k - k_0, C_{\text{CR1}} := (n+1)L + n\bar{\kappa} + 2nL_H R, C_{\text{CR2}} := 2nL_H C_0, C_0 := \sqrt{\frac{3r_{k_0}^2}{2} + M(\varphi(F(x_{k_0}) - \bar{F}))}, M := \frac{3D}{L_H}, D = (2n\bar{\kappa} + 2L + 2L_H R), C_1 := 3\varphi(F(x_{k_0}) - \bar{F})$  and  $R = (6(F(x_0) - \inf F)/L_H)^{1/3}$ . Furthermore, for the case that  $\varphi$  has the form  $\varphi(t) = ct^{1-\theta}$  for some  $c > 0$ , Cubic SR1 PQN has better convergence rates:

1. If  $\theta \in (0, \frac{1}{2})$ , it has a superlinear convergence rate:

$$\|F'(x_{N+k_0})\| \leq \left( C \left( \frac{C_{\text{CR1}}}{N} + \frac{C_{\text{CR2}}}{N^{1/2}} \right) \right)^{N/2} \|F'(x_{k_0})\|; \quad (9)$$

2. if  $\theta = \frac{1}{2}$ , it has a superlinear convergence rate:

$$\|F'(x_{N+k_0})\| \leq \left( \frac{3c^2}{4} \left( \frac{C_{\text{CR1}}}{N} + \frac{C_{\text{CR2}}}{N^{1/2}} \right) \right)^{N/2} \|F'(x_{k_0})\|; \quad (10)$$

3. if  $\theta \in (\frac{1}{2}, 1)$ ,

$$\min_{\{i \in \mathbb{N} | k_0 \leq i \leq N+k_0\}} \|F'(x_i)\| \leq \left( C_2 \left( \frac{C_{\text{CR1}}}{N} + \frac{C_{\text{CR2}}}{N^{1/2}} \right) \|F'(x_{k_0})\|^{1/(\theta N)} \right)^{N/(2+(N-1)(2-\frac{1}{\theta}))}, \quad (11)$$

where  $C_2 = 3((1-\theta)c)^{1/\theta}$  and  $C = C_1^{\frac{1-2\theta}{1-\theta}} C_2^{\frac{\theta}{1-\theta}}$ .

*Proof.* See Section 5.2.  $\square$

**Remark 4.3.** If  $F$  is coercive, then  $(x_k)_{k \in \mathbb{N}}$  is bounded automatically, since  $f(x_k)$  decreases monotonically and  $(x_k)_{k \in \mathbb{N}}$  is guaranteed to stay inside  $[f \leq f(x_0)]$  (See Lemma 44).

**Remark 4.4.** (9) and (10) do not contain any “min”, which can be understood as the fast growth rate of the function ensuring a sufficient decrease of  $\|\nabla f(x_k)\|$  when  $\theta \in (0, 1/2]$ .

**Remark 4.5.** It is not necessary for  $x_{k+1}$  to be the global minimum point in the update step 1. To obtain the same theoretical results, it requires a stationary point  $x_{k+1}$  satisfying the following inequality:

$$g(x_{k+1}) + \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \frac{1}{2} \|x_{k+1} - x_k\|_{G_k + L_H r_{k-1} I} + \frac{L_H}{3} \|x_{k+1} - x_k\|^3 \leq g(x_k). \quad (12)$$

**Remark 4.6.** The convergence analysis of Theorem 4.2 relies on the uniformized KL property of  $x$  in Lemma 3.1. We can obtain global convergence rates if the KL property is global, i.e. if (3) is satisfied with  $\epsilon = +\infty$  and  $\eta = +\infty$ . For instance, the case where  $F$  verifies the global Łojasiewicz inequality with  $\theta = 1/2$  has become recently popular in the theory of neural network training. In particular when  $g = 0$ , we have  $F = f$  and Algorithm 1 exhibits a non-asymptotic superlinear convergence rate:

$$\|\nabla f(x_N)\| \leq \left( C \left( \frac{\overline{C}_{\text{CR1}}}{N} + \frac{\overline{C}_{\text{CR2}}}{N^{1/2}} \right) \right)^{N/2} \|\nabla f(x_0)\|. \quad (13)$$

where  $\overline{C}_{\text{CR1}} := (n+1)L + n\kappa + 2nL_H R$ ,  $\overline{C}_{\text{CR2}} := 2nL_H C_0$ ,  $C_0 := \sqrt{\frac{3r_{k_0}^2}{2} + M(\sqrt{f(x_0) - \min f})}$ ,  $M := \frac{3D}{L_H}$ ,  $D = (n\bar{\kappa} + 2L + 2L_H R)$ ,  $C_1 := 3\sqrt{f(x_0) - \min f}$  and  $R = (6(f(x_0) - \min f)/L_H)^{1/3}$ .

In fact, a slight modification of the proof of Theorem 4.2 shows that even without using a restarting strategy, the following refined superlinear convergence rate can be obtained for Algorithm 3 (see Appendix A):

$$\|\nabla f(x_N)\| \leq \frac{\mu}{6} \left( \frac{\Theta}{N^{1/3}} \right)^{N/2} \|\nabla f(x_0)\|, \quad (14)$$

where  $\Theta = \left( 2nL + 2nL_H R + 2nL_H \left( \frac{6}{L_H} (f(x_0) - \inf f) \right)^{1/3} \right)$  and  $R = (6(f(x_0) - \min f)/L_H)^{1/3}$ .

**Remark 4.7.** When  $\varphi(t) = ct^{1/2}$ , (10) indicates that the super-linear rate is attained after  $N+k_0 \geq \max\{C_{\text{CR1}}, C_{\text{CR2}}^2\} + k_0$  number of iterations. Here  $C_{\text{CR1}} = O(n)$  and  $C_{\text{CR2}} = O(n^{3/2})$  where  $n$  is the number of dimension. Therefore, the super-linear rate is attained after  $\Omega(n^3) + k_0$  iterations.

**Remark 4.8.** We compare our results with convergence rates of first-order methods. When  $N$  is sufficiently large, for a general  $\varphi$ , the Cubic SR1 PQN has a local sublinear convergence rate  $O(\frac{1}{N^{1/2}})$ . Now, we consider the case when  $\varphi = ct^{1-\theta}$ . For  $\theta \in (0, \frac{1}{2}]$ , first-order methods achieve linear convergence rates [38] while our Cubic SR1 method attains a super-linear convergence rate.

When  $\theta \in (\frac{1}{2}, 1)$ , the first-order methods have sublinear convergence rates  $O\left(\frac{1}{N^{\frac{1-\theta}{2\theta-1}}}\right)$  (see [1]), whereas our Cubic SR1 method has a convergence rate close to  $O\left(\frac{1}{N^{\frac{1}{2(2\theta-1)}}}\right)$ . For  $\frac{1}{2} < \theta < \frac{2}{3}$ , the convergence rate of our Cubic SR1 PQN is slower than that of first-order methods [26], but for  $\frac{2}{3} < \theta \leq 1$  our Cubic SR1 PQN achieves a faster convergence rate. This phenomenon appears because if a function is too flat, then (close to) degeneracy of the Hessian seems to penalize SR1 and thus does not bring any additional benefit.

#### 4.2.2 Convex objectives

If we additionally assume that  $f$  and  $g$  are convex functions, then  $F$  is convex and we can apply a gradient regularized quasi-Newton method. That is Algorithm 2 which is the same as the third algorithm proposed in [54]. We recap the algorithm here for the readers' convenience.

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#### Algorithm 2 Grad SR1 PQN

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**Require:**  $x_0, G_0 = LI, \lambda_0 = 0, \tilde{G}_0 = G_0 + \lambda_0 I, \bar{\kappa} \geq L$ .

**Update for**  $k = 0, \dots, N$ :

1. **Update**

$$x_{k+1} = \underset{x \in \mathbb{R}^n}{\operatorname{argmin}} g(x) + \langle \nabla f(x_k), x - x_k \rangle + \frac{1}{2} \|x - x_k\|_{\tilde{G}_k}^2. \quad (15)$$

2. **Denote but do not use explicitly**  $J_k := \int_0^1 \nabla^2 f(x_k + tu_k) dt$ .

3. **Compute**  $u_k = x_{k+1} - x_k, r_k = \|u_k\|$ , and

$$\begin{aligned} F'(x_{k+1}) &= \nabla f(x_{k+1}) - \nabla f(x_k) - \tilde{G}_k u_k, \\ G_{k+1} &= \operatorname{SR1}(J_k, \tilde{G}_k, u_k), \\ \lambda_{k+1} &= \left( \sqrt{L_H \|F'(x_{k+1})\|} + L_H r_k \right). \end{aligned}$$

4. **Update**  $\tilde{G}_{k+1}$ : compute  $\hat{G}_{k+1} = G_{k+1} + \lambda_{k+1} I$  (**Correction step**).

**If**  $\operatorname{tr} \hat{G}_{k+1} \leq n\bar{\kappa}$ : we set  $\tilde{G}_{k+1} = \hat{G}_{k+1}$ .

**Othwise:** we set  $\tilde{G}_{k+1} = LI$  (**Restart step**).

5. **If**  $\|F'(x_{k+1})\| = 0$ : Terminate.

**End**

---

For gradient regularized proximal quasi-Newton methods, we obtain global convergence with the following non-asymptotic rates.

**Theorem 4.9** (Convex objectives). *Let Assumption 1 and 2 hold. Assume that the sequence  $(x_k)_{k \in \mathbb{N}}$  generated by Grad SR1 PQN is bounded, i.e. there exists  $R$  such that  $R \geq 2\|x_k\|$  for any  $k \in \mathbb{N}$ . Additionally, we assume both  $g, f$  are convex. For any initialization  $x_0 \in \mathbb{R}^n$  Grad SR1 PQN (Algorithm 2) has  $\|F'(x_k)\| \rightarrow 0, F(x_k) \rightarrow \min F$  and  $\operatorname{dist}(x_k, \arg \min F) \rightarrow 0$  as  $k \rightarrow \infty$ . Moreover, there exists  $k_0 \in \mathbb{N}$  such that for any  $k \geq k_0$ , Grad SR1 PQN (Algorithm 2) has a local non-asymptotic rate:*

$$\left( \min_{\{i | k_0 \leq i \leq N+k_0\}} \|F'(x_i)\| \right) \leq \left( S_1 \left( \frac{n\bar{\kappa}}{N} + \frac{C_{\text{GR}}}{N^{1/3}} \right) \right)^{N/(N+1)} \|F'(x_{k_0})\|^{1/(N+1)}; \quad (16)$$

where  $N = k - k_0, C_{\text{GR}} := nDC_0^{1/3}, C_0 := \frac{r_{k_0}^{3/2}}{3} + \sqrt{\frac{4(n\bar{\kappa}+L)}{L_H}}(\varphi(F(x_{k_0}) - \min F)), D := (\sqrt{L_H(n\bar{\kappa}+L)} + L_H R)$ , and  $S_1 = 2\varphi(F(x_{k_0}) - \min F)$ .

Furthermore, for the case  $\varphi$  has the form  $\varphi(t) = ct^{1-\theta}$  for some  $c > 0$ , Grad SR1 PQN has convergence rates:

1. if  $\theta \in (0, \frac{1}{2})$ , it has a super-linear convergence rate:

$$\|F'(x_{N+k_0})\| \leq \left( S \left( \frac{n\bar{\kappa}}{N} + \frac{C_{\text{GR}}}{N^{1/3}} \right) \right)^{N/2} \|F'(x_{k_0})\|, \quad (17)$$



2. if  $\theta = \frac{1}{2}$ , it has a super-linear convergence rate:

$$\|F'(x_{N+k_0})\| \leq \left( \frac{c^2}{2} \left( \frac{n\bar{\kappa}}{N} + \frac{C_{\text{GR}}}{N^{1/3}} \right) \right)^{N/2} \|F'(x_{k_0})\|, \quad (18)$$

3. if  $\theta \in (\frac{1}{2}, 1)$ , it has a sub-linear convergence rate:

$$\min_{\{i \in \mathbb{N} | k_0 \leq i \leq N+k_0\}} \|F'(x_i)\| \leq \left( S_2 \left( \frac{n\bar{\kappa}}{N} + \frac{C_{\text{GR}}}{N^{1/3}} \right) \|F'(x_{k_0})\|^{1/(\theta N)} \right)^{1/(\frac{2}{N} + (N-1)(2-\frac{1}{\theta}))}, \quad (19)$$

$$\text{where } S_2 = 2((1-\theta)c)^{1/\theta} \text{ and } S = S_1^{\frac{1-2\theta}{1-\theta}} S_2^{\frac{\theta}{1-\theta}}.$$

*Proof.* See Section 5.3.  $\square$

**Remark 4.10.**  $(x_k)_{k \in \mathbb{N}}$  is bounded under coercivity of  $F$ .

**Remark 4.11.** The convergence analysis of Theorem 4.9 also relies on the uniformized KL property of  $F$  with respect to  $\arg \min F$ . (16) can be made global if  $F$  verifies a global version of the KL inequality, i.e. with  $\epsilon = +\infty$  and  $\eta = +\infty$ .

**Remark 4.12.** In fact, in the case where  $\varphi(t) = ct^{1/2}$ , (10) indicates that the super-linear rate is attained after  $N + k_0 \geq \max\{n\bar{\kappa}, C_{\text{GR}}^3\} + k_0$  number of iterations and  $C_{\text{GR}} = O(n^{5/3})$ . Therefore, the super-linear rate is attained after  $\Omega(n^5) + k_0$  iterations.

Let  $g = 0$ . Let  $f$  satisfy Assumption 1 and the gradient domination inequality (global Lojasiewicz inequality):

$$f(x) - \min f \leq \frac{c}{2} \|\nabla f(x)\|^2, \quad (20)$$

for some  $c > 0$  and any  $x$ . A better global convergence rate can be retrieved.

**Theorem 4.13.** Let Assumption 1 and 2 hold. Additionally, we assume  $g = 0$  and  $f$  is a convex function satisfying (20). For any initialization  $x_0 \in \mathbb{R}^n$  Grad SR1 PQN (Algorithm 2) has  $\|\nabla f(x_N)\| \rightarrow 0$  as  $N \rightarrow +\infty$  and Grad SR1 PQN (Algorithm 2) has a global non-asymptotic rate:

$$\|\nabla f(x_N)\| \leq \left( \frac{c^2 C_{\text{GD}}}{2N} \right)^{N/(2)} \|\nabla f(x_0)\|^{2/(N+1)}; \quad (21)$$

where  $C_{\text{GD}} := (n\bar{\kappa} + n\sqrt{L_H}C_f + nL_H C_r)$ ,  $C_f = 4n\bar{\kappa}(cn\bar{\kappa}(f(x_0) - \inf f))^{1/4}$  and

$$C_r = \frac{1}{\sqrt{L_H}} 4cn\bar{\kappa}(2n\bar{\kappa}(f(x_0) - \inf f))^{1/4} + \frac{1}{L} 2cn\bar{\kappa}(2n\bar{\kappa}(f(x_0) - \inf f))^{1/2}.$$

*Proof.* See Section 5.3.  $\square$

**Remark 4.14.** (21) indicates that the super-linear rate is attained after  $N \geq C_{\text{GD}}$  number of iterations and  $C_{\text{GD}} = O(n^{2.5})$ .

## 5 Convergence Analysis

In this section, without loss of generality, we assume that  $\|F'(x_k)\| > 0$  for any  $k \in \mathbb{N}$ , since, otherwise, our algorithms terminate after a finite number of steps and our non-asymptotic rates would hold until then. For convenience, we denote  $F'(x_0)$  as any subgradient of  $F(x)$  at  $x_0$ .

Recall that throughout the whole convergence analysis, we assume that Assumption 1 and 2 hold with all variables as defined in Algorithm 1.



## 5.1 Preparatory lemmas

First, we need several important properties of  $J_k := \int_0^1 \nabla^2 f(x_k + tu_k) dt$  on which we will capitalize for the analysis of both algorithms.

**Lemma 5.1.** *For each  $k \in \mathbb{N}$ , we have*

$$f(x_{k+1}) - f(x_k) \leq \langle \nabla f(x_k), u_k \rangle + \frac{1}{2} \langle \nabla^2 f(x_k) u_k, u_k \rangle + \frac{L_H}{6} \|u_k\|^3, \quad (22)$$

$$f(x_{k+1}) - f(x_k) \leq \langle \nabla f(x_k), u_k \rangle + \frac{1}{2} \langle J_k u_k, u_k \rangle + \frac{L_H}{6} \|u_k\|^3. \quad (23)$$

*Proof.* The first result is from [42, Lemma 1]. Using a similar argument as in [42, Lemma 1], we can derive the second result. For any  $y, x \in \mathbb{R}^n$  and  $t \in (0, 1]$ , we have

$$\begin{aligned} & \|\nabla f(x + t(y - x)) - \nabla f(x) - \int_0^1 \nabla^2 f(x + s(y - x)) t(y - x) ds\| \\ & \leq \left\| \int_0^1 (\nabla^2 f(x + st(y - x)) - \nabla^2 f(x + s(y - x))) t(y - x) ds \right\| \\ & \leq L_H \|y - x\|^2 \int_0^1 |(s - st)t| ds \\ & \leq \frac{(1-t)t}{2} L_H \|y - x\|^2. \end{aligned} \quad (24)$$

Therefore, we have

$$\begin{aligned} |f(x_{k+1}) - f(x_k) - \langle \nabla f(x_k), u_k \rangle - \frac{1}{2} \langle J_k u_k, u_k \rangle| & \leq \left| \int_0^1 \langle \nabla f(x_k + tu_k) - \nabla f(x_k) - tJ_k u_k, u_k \rangle dt \right| \\ & \leq \|u_k\| \int_0^1 \|\nabla f(x_k + tu_k) - \nabla f(x_k) - tJ_k u_k\| dt \\ & \leq \frac{L_H}{2} \|u_k\|^3 \int_0^1 (t - t^2) dt \\ & \leq \frac{L_H}{6} \|u_k\|^3. \end{aligned} \quad (25)$$

□

**Lemma 5.2.** *For each  $k \in \mathbb{N}$ , we have  $-LI \preceq J_k \preceq LI$ .*

*Proof.* Since  $f$  has  $L$ -Lipschitz gradient, we have  $-LI \preceq \nabla^2 f(x) \preceq LI$  for any  $x \in \mathbb{R}^n$ . Thus, due to the definition of  $J_k$ , we have  $-LI \preceq J_k \preceq LI$ . □

**Lemma 5.3.** *For each  $k \in \mathbb{N}$ , we have*

$$J_k \preceq J_{k-1} + \frac{1}{2} L_H (r_k + r_{k-1}) I, \quad (26)$$

$$\nabla^2 f(x_k) \preceq J_{k-1} + \frac{1}{2} L_H r_{k-1} I, \quad (27)$$

$$\nabla^2 f(x_k) \preceq J_k + \frac{1}{2} L_H r_k I. \quad (28)$$

*Proof.* The proof is similar to the one of [49, Lemma 4.2], [56, Lemma 5]. The assumption that the Hessian of  $f$  is  $L_H$ -Lipschitz continuous means that there exists  $L_H > 0$  such that for any  $x, y \in \mathbb{R}^n$ , we have

$$\|\nabla^2 f(x) - \nabla^2 f(y)\| \leq L_H \|x - y\|. \quad (29)$$

For any  $t, s \in [0, 1]$  and  $k \in \mathbb{N}$ , letting  $x = x_k + tu_k$  and  $y = x_k - (1-s)u_{k-1}$  in (29), we have that

$$\|\nabla^2 f(x_k + tu_k) - \nabla^2 f(x_k - (1-s)u_{k-1})\| \leq L_H \|tu_k + (1-s)u_{k-1}\|. \quad (30)$$

Then, by the definition of  $J_k$ , we have

$$\begin{aligned}
J_k - J_{k-1} &= \int_0^1 \nabla^2 f(x_k + tu_k) dt - \int_0^1 \nabla^2 f(x_{k-1} + su_{k-1}) ds \\
&= \int_0^1 \nabla^2 f(x_k + tu_k) dt - \int_0^1 \nabla^2 f(x_k - (1-s)u_{k-1}) ds \\
&\preceq \int_0^1 \|\nabla^2 f(x_k + su_k) - \nabla^2 f(x_k - (1-s)u_{k-1})\| ds \cdot I \\
&\preceq \int_0^1 L_H \|su_k + (1-s)u_{k-1}\| dt \cdot I \\
&\preceq \frac{1}{2} L_H (r_k + r_{k-1}) I,
\end{aligned} \tag{31}$$

where the first inequality holds because of the definition of the induced matrix norm, the second inequality holds due to (30) and the last one uses the triangle inequality. Using the same trick, we have

$$\begin{aligned}
\nabla^2 f(x_k) - J_{k-1} &= \int_0^1 \nabla^2 f(x_k) dt - \int_0^1 \nabla^2 f(x_{k-1} + su_{k-1}) ds \\
&\preceq \int_0^1 \|\nabla^2 f(x_k) - \nabla^2 f(x_k - (1-s)u_{k-1})\| ds \cdot I \\
&\preceq \int_0^1 L_H \|(1-s)u_{k-1}\| dt \cdot I \\
&\preceq \frac{1}{2} L_H r_{k-1} I.
\end{aligned} \tag{32}$$

We argue similarly to obtain (28).  $\square$

We now recall an important property of the SR1 method, which shows that it preserves the partial order of matrices.

**Lemma 5.4.** *For any two symmetric matrices  $A \in \mathbb{R}^{n \times n}$  and  $G \in \mathbb{R}^{n \times n}$  with  $A \preceq G$  and any  $u \in \mathbb{R}^n$ , we have the following for  $G_+ = \text{SR1}(A, G, u)$ :*

$$A \preceq G_+ \preceq G. \tag{33}$$

*Proof.* The argument is similar to the one in [49, Part of Lemma 2.2] despite they assume  $A$  and  $G$  are positive definite. We notice that the result holds true for general symmetric matrices  $A$  and  $G$  with  $A \preceq G$ . Let  $u$  be given. If  $(G - A)u = 0$ ,  $G_+ = G$ . We now turn to the case when  $(G - A)u \neq 0$ . This means, since  $G - A \succeq 0$ , that  $u^\top (G - A)u > 0$  for given  $u \in \mathbb{R}^n$ , and there exists a decomposition  $G - A = VV^\top$  for some  $V$ . Then, we have

$$\begin{aligned}
G_+ - A &= G - A - \frac{(G - A)uu^\top(G - A)}{u^\top(G - A)u} \\
&= VV^\top - V \frac{\tilde{u}\tilde{u}^\top}{\tilde{u}^\top \tilde{u}} V^\top \\
&= V(I - \frac{\tilde{u}\tilde{u}^\top}{\tilde{u}^\top \tilde{u}})V^\top \succeq 0,
\end{aligned} \tag{34}$$

where  $\tilde{u} = V^\top u$ . Recall that since  $u^\top (G - A)u > 0$ ,  $\tilde{u} \neq 0$  for the given  $u \neq 0$ , and thus (34) makes sense. Thus,  $G_+ \succeq A$ . A similar argument also proves that  $G_+ \preceq G$ .  $\square$

Following [54], for any symmetric matrix  $G$ , we introduce the potential function:

$$V(G) := \text{tr } G, \tag{35}$$

and the following function:

$$\nu(A, G, u) = \begin{cases} 0, & \text{if } (G - A)u = 0; \\ \frac{u^\top (G - A)(G - A)u}{u^\top (G - A)u}, & \text{otherwise.} \end{cases} \tag{36}$$

Using the potential function  $V(G)$  and  $\nu(A, G, u)$ , the following lemma shows that the SR1 update leads to a better approximation of  $A$ .

**Lemma 5.5.** Consider two symmetric matrices  $A \in \mathbb{R}^{n \times n}$  and  $G \in \mathbb{R}^{n \times n}$  with  $A \preceq G$  and any  $u \in \mathbb{R}^n$ . Let  $G_+ = \text{SR1}(A, G, u)$ . Then, we have

$$V(G) - V(G_+) = \nu(A, G, u), \quad (37)$$

*Proof.* The argument is the same as the one in [54, Lemma 2.4] where they assumed  $A$  and  $G$  are positive definite. However, the argument holds true for symmetric matrices  $A$  and  $G$ .  $\square$

## 5.2 Convergence analysis of Algorithm 1

Now, let us analyze Algorithm 1. Since Step 1 has two cases, we have to investigate the update step in (6) and the one in (7) separately. The necessary optimality condition of the update step in (6), when  $\text{tr } G_k \leq n\bar{\kappa}$ , implies

$$\nabla f(x_k) + \partial g(x_{k+1}) + (G_k + L_H(r_{k-1} + r_k)I)(x_{k+1} - x_k) \ni 0. \quad (38)$$

In this case, we set  $\tilde{G}_{k+1} = G_k + \lambda_k I$  and  $\lambda_k = L_H(r_{k-1} + r_k)$ . The necessary optimality condition of the update step in (7), when  $\text{tr } G_k > n\bar{\kappa}$ , is

$$\nabla f(x_k) + \partial g(x_{k+1}) + (L + L_H r_{k-1} + L_H r_k)(x_{k+1} - x_k) \ni 0. \quad (39)$$

In this case, we set  $\tilde{G}_{k+1} = (L + \lambda_k)I$ . In both cases, though the definitions of  $\tilde{G}_k$  are different, (38) and (39) read:

$$\nabla f(x_k) + v_{k+1} + \tilde{G}_{k+1}(x_{k+1} - x_k) = 0, \quad (40)$$

where  $v_{k+1} \in \partial g(x_{k+1})$ .

We denote  $F'(x_k) := v_k + \nabla f(x_k)$ . Thus,  $F'(x_k) \in \partial F(x_k)$ .

**Lemma 5.6.** For each  $k \in \mathbb{N}$ , we have

$$J_k \preceq G_{k+1} \preceq \tilde{G}_{k+1}, \quad (41)$$

$$V(G_{k+1}) \leq V(\tilde{G}_{k+1}). \quad (42)$$

*Proof.* We prove the inequalities in (41) by induction. In order to validate the base case  $k = 0$ , we first observe that  $\tilde{G}_1 = G_0 + \lambda_0 I \succeq J_0$  holds by Lipschitz continuity of  $\nabla f$  and  $G_0 = LI$  and  $\lambda_0 > 0$ . Then, Lemma 5.4 shows the desired property:

$$J_0 \preceq G_1 = \text{SR1}(J_0, \tilde{G}_1, u_0) \preceq \tilde{G}_1,$$

and for showing the induction, we suppose now that (41) holds for  $k - 1$ . We discuss two cases separately:

1. If  $\text{tr } G_k \leq n\bar{\kappa}$ , then,  $\tilde{G}_{k+1} = G_k + \lambda_k I$ . According to the induction hypothesis and Lemma 5.3, we have

$$\begin{aligned} \tilde{G}_{k+1} &= G_k + \lambda_k I \\ &\succeq J_{k-1} + \lambda_k I \\ &= J_{k-1} + (L_H r_{k-1} + L_H r_k)I \\ &\succeq J_k. \end{aligned} \quad (43)$$

We are then in position to apply Lemma 5.4 again. As a result, we obtain  $J_k \preceq G_{k+1} = \text{SR1}(J_k, \tilde{G}_{k+1}, u_k) \preceq \tilde{G}_{k+1}$  and  $\text{tr } G_{k+1} \leq \text{tr } \tilde{G}_{k+1}$ .

2. If  $\text{tr } G_k > n\bar{\kappa}$ , then  $\tilde{G}_{k+1} = (L + \lambda_k)I$  due to the restarting step. Using Lemma 5.2, we have  $\tilde{G}_{k+1} \succeq J_k$  and using Lemma 5.4 again, we obtain  $J_k \preceq G_{k+1} \preceq \tilde{G}_{k+1}$  and thus,  $\text{tr } G_{k+1} \leq \text{tr } \tilde{G}_{k+1}$ .  $\square$

Based on the lemma above, we can show the monotone decrease of the function value.

**Lemma 5.7.** For each  $k \in \mathbb{N}$ , we have

$$F(x_{k+1}) - F(x_k) \leq -\frac{L_H}{6} r_k^3, \quad (44)$$

$$\langle (\tilde{G}_{k+1} - J_k)u_k, u_k \rangle \leq 2(F(x_k) - F(x_{k+1})) + \frac{2L_H}{3} r_k^3. \quad (45)$$

*Proof.* We prove again each case separately. If  $\text{tr } G_k \leq n\bar{\kappa}$ , the optimality of  $x_{k+1}$  in (6) implies that

$$g(x_{k+1}) + \langle \nabla f(x_k), u_k \rangle + \frac{1}{2} \langle (G_k + L_H r_{k-1}) u_k, u_k \rangle + \frac{L_H}{3} r_k^3 \leq g(x_k). \quad (46)$$

This together with Lemma 5.1 yields

$$\begin{aligned} F(x_{k+1}) &\leq F(x_k) - \frac{1}{2} \langle (G_k + L_H r_{k-1} - \nabla^2 f(x_k)) u_k, u_k \rangle - \frac{L_H}{6} r_k^3 \\ &\leq F(x_k) - \frac{1}{2} \left\langle \left( G_k + L_H r_{k-1} - J_{k-1} - \frac{1}{2} L_H r_{k-1} \right) u_k, u_k \right\rangle - \frac{L_H}{6} r_k^3 \\ &\leq F(x_k) - \frac{L_H}{6} r_k^3, \end{aligned} \quad (47)$$

where the second inequality holds due to Lemma 5.3 and the last inequality holds due to Lemma 5.6. Similarly, thanks to Lemma 5.1, we have

$$\begin{aligned} F(x_{k+1}) &\leq F(x_k) - \frac{1}{2} \langle (G_k + L_H r_{k-1} - J_k) u_k, u_k \rangle - \frac{L_H}{6} r_k^3 \\ &= F(x_k) - \frac{1}{2} \langle (G_k + L_H r_{k-1} + L_H r_k - J_k - L_H r_k) u_k, u_k \rangle - \frac{L_H}{6} r_k^3 \\ &\leq F(x_k) - \frac{1}{2} \langle (\tilde{G}_{k+1} - J_k) u_k, u_k \rangle + \frac{L_H}{3} r_k^3. \end{aligned} \quad (48)$$

Let's now focus on the case where  $\text{tr } G_k > n\bar{\kappa}$ . The update (7) implies that

$$g(x_{k+1}) + \langle \nabla f(x_k), u_k \rangle + \frac{1}{2} \langle (L + L_H r_{k-1}) u_k, u_k \rangle + \frac{L_H}{3} r_k^3 \leq g(x_k), \quad (49)$$

since  $x_{k+1}$  is the minimum of the subproblem. With Lemma 5.1, we obtain

$$\begin{aligned} F(x_{k+1}) &\leq F(x_k) - \frac{1}{2} \langle ((L + L_H r_{k-1})I - \nabla^2 f(x_k)) u_k, u_k \rangle - \frac{L_H}{6} r_k^3 \\ &\leq F(x_k) - \frac{L_H}{6} r_k^3, \end{aligned} \quad (50)$$

where the last inequality holds since  $L \succeq \nabla^2 f(x_k)$ . Similarly, we have

$$\begin{aligned} F(x_{k+1}) &\leq F(x_k) - \frac{1}{2} \langle ((L + L_H r_{k-1})I - J_k) u_k, u_k \rangle - \frac{L_H}{6} r_k^3 \\ &\leq F(x_k) - \frac{1}{2} \langle ((L + L_H r_{k-1} + L_H r_k)I - J_k - L_H r_k) u_k, u_k \rangle - \frac{L_H}{6} r_k^3 \\ &\leq F(x_k) - \frac{1}{2} \langle (\tilde{G}_{k+1} - J_k) u_k, u_k \rangle + \frac{L_H}{3} r_k^3. \end{aligned} \quad (51)$$

□

Due to the decrease of the function values, we have for any  $k$ ,  $r_k \leq \frac{F(x_0) - \inf F}{L_H}$ . Thus, for any  $k \in \mathbb{N}$ ,  $r_k \leq R$  where  $R = (6(F(x_0) - \inf F)/L_H)^{1/3}$ .

**Lemma 5.8.** *For each  $k \in \mathbb{N}$ , we have  $\|\tilde{G}_{k+1}\| \leq 2n\bar{\kappa} + L + 2L_H R$ , where  $R = (6(F(x_0) - \inf F)/L_H)^{1/3}$ .*

*Proof.* Case  $\text{tr } G_k \leq n\bar{\kappa}$ : we have  $\tilde{G}_{k+1} = G_k + \lambda_k I = G_k + L_H(r_{k-1} + r_k)I$ . Thus, using Lemma 5.2 and Lemma 5.6, we get  $\|\tilde{G}_{k+1}\| \leq \|\tilde{G}_{k+1} - J_{k-1}\| + \|J_{k-1}\| \leq \text{tr}(G_k - J_{k-1}) + L + L_H r_k + L_H r_{k-1} \leq n\bar{\kappa} + (n+1)L + L_H r_k + L_H r_{k-1} \leq 2n\bar{\kappa} + L + 2L_H R$  where  $R = (6(F(x_0) - \inf F)/L_H)^{1/3}$ .

Case  $\text{tr } G_k > n\bar{\kappa}$ : we have  $\tilde{G}_{k+1} = LI + \lambda_k I$ . Thus, we have  $\|\tilde{G}_{k+1}\| \leq L + L_H r_k + L_H r_{k-1} \leq 2L + 2L_H R \leq 2n\bar{\kappa} + L + 2L_H R$ . □

Let  $\omega(x_0)$  be the set of clusters of  $x_k$  generated by Algorithm 1.

**Lemma 5.9.** *Assume  $(x_k)_{k \in \mathbb{N}}$  generated by Algorithm 1 is bounded. The following holds:*

1.  $\|F'(x_k)\| \rightarrow 0$  and  $F(x_k) \rightarrow \bar{F}$  as  $k \rightarrow \infty$ ,

2.  $\omega(x_0) \subset \text{crit} F$ ,

3. the objective function  $F(x) \equiv \bar{F}$  is constant over  $\omega(x_0)$  where  $\bar{F} \in \mathbb{R}$ , and

*Proof.* This proof is similar to that of [10, Lemma 5]. Thanks to Lemma 5.7 and  $\inf F$  is finite (Assumption 1), we can deduce that  $r_k \rightarrow 0$  as  $k \rightarrow \infty$  and  $F(x_k) \rightarrow \bar{F}$  for some  $\bar{F} \in \mathbb{R}$ . By (40), we have

$$F'(x_{k+1}) = \nabla f(x_{k+1}) - \nabla f(x_k) - \tilde{G}_{k+1}(x_{k+1} - x_k) \in \partial F(x_{k+1}). \quad (52)$$

Since  $\nabla f$  is Lipschitz continuous,  $r_k \rightarrow 0$  as  $k \rightarrow +\infty$  and  $\tilde{G}_k$  is bounded, we have  $\|F'(x_{k+1})\| \rightarrow 0$  as  $k \rightarrow +\infty$ , proving the first part of 1. Since the sequence  $(x_k)_{k \in \mathbb{N}}$  is bounded, there exists a subsequence  $(x_{k_q})_{q \in \mathbb{N}}$  that converges to some  $\bar{x}$ . Since (52) holds true for any subsequences, we get  $\|F'(x_{k_q})\| \rightarrow 0$  as  $q \rightarrow \infty$ . Letting  $q \rightarrow \infty$ , by the closedness of the graph of  $\partial F$ , we deduce that  $\partial F(\bar{x}) \ni 0$ . For arbitrary  $\bar{x} \in \omega(x_0)$ , there exists a convergent subsequence  $(x_{k_q})_{q \in \mathbb{N}}$  and  $\partial F(\bar{x}) \ni 0$ . This shows 2. Since  $x_{k_q} \rightarrow \bar{x}$  as  $q \rightarrow \infty$  and  $F(x_k) \rightarrow \bar{F}$  as  $k \rightarrow \infty$ ,  $F(x_{k_q}) \rightarrow F(\bar{x}) = \bar{F}$  as  $q \rightarrow \infty$ . We recall that  $\bar{x}$  is an arbitrary cluster. Therefore,  $F$  is constant over  $\omega(x_0)$ .  $\square$

**Lemma 5.10.** For any  $k \in \mathbb{N}$ , we have  $\|F'(x_{k+1})\| \leq Dr_k$  where  $D = (2n\bar{\kappa} + 2L + 2L_H R)$  and  $R = (6(F(x_0) - \inf F)/L_H)^{1/3}$ .

*Proof.* By Lemma 5.8, we have

$$\begin{aligned} \|F'(x_{k+1})\| &= \|v_{k+1} + \nabla f(x_{k+1})\| \\ &= \|\nabla f(x_{k+1}) - \nabla f(x_k) - \tilde{G}_{k+1}(x_{k+1} - x_k)\| \\ &\leq \|\nabla f(x_{k+1}) - \nabla f(x_k)\| + \|\tilde{G}_{k+1}(x_{k+1} - x_k)\| \\ &\leq Lr_k + (2n\bar{\kappa} + L + 2L_H R)r_k \\ &\leq (2n\bar{\kappa} + 2L + 2L_H R)r_k. \end{aligned} \quad (53)$$

$\square$

Next, we study the growth rates of the sequences  $r_k$  and  $\lambda_k$ .

**Lemma 5.11.** Given a number of iteration  $N \in \mathbb{N}$ , for large enough  $k_0$ , we have:

$$\sum_{k=k_0}^{N-1+k_0} r_k \leq C_0 N^{1/2}, \quad (54)$$

$$\sum_{k=k_0+1}^{N+k_0} \lambda_k \leq 2C_0 L_H N^{1/2}, \quad (55)$$

where  $C_0 := \sqrt{\frac{3r_{k_0}^2}{2} + M(\varphi(F(x_{k_0}) - \bar{F}))}$  and  $M := \frac{3D}{L_H}$ .

*Proof.* Thanks to Assumption 1, 2 and Lemma 3.1,  $F$  has the uniformized KL property over  $\omega(x_0)$ , i.e., there exists  $\epsilon > 0$ ,  $\eta > 0$  and  $\varphi \in \Phi_\eta$  such that for all  $\bar{x} \in \omega(x_0)$  and all  $x$  in the following intersection

$$\{x \in \mathbb{R}^n | \text{dist}(x, \Omega) < \epsilon\} \cap [F(\bar{x}) < F < F(\bar{x}) + \eta], \quad (56)$$

one has,

$$\varphi'(F(x) - F(\bar{x})) \text{dist}(0, \partial F(x)) \geq 1. \quad (57)$$

Moreover, in view of Lemma 5.9, there exists some  $k_0$  such that for any  $k \geq k_0$ ,

$$x_k \in \{x \in \mathbb{R}^n | \text{dist}(x, \omega(x_0)) < \epsilon\} \cap [\bar{F} < F < \bar{F} + \eta],$$

where  $\bar{F} = F(\bar{x})$ .

We start with the first two inequalities. Using Lemma 5.7, we have:

$$\begin{aligned} \varphi(F(x_k) - \bar{F}) - \varphi(F(x_{k+1}) - \bar{F}) &\geq \varphi'(F(x_k) - \bar{F})(F(x_k) - F(x_{k+1})) \\ &\geq \frac{F(x_k) - F(x_{k+1})}{\text{dist}(0, \partial F(x_k))} \geq \frac{F(x_k) - F(x_{k+1})}{\|F'(x_k)\|} \\ &\geq \frac{\frac{1}{6}L_H r_k^3}{\|F'(x_k)\|} \\ &\geq \frac{\frac{1}{6}L_H r_k^3}{Dr_{k-1}} \end{aligned} \quad (58)$$

where the first inequality uses the concavity of  $\varphi$ , the second one uses the uniformized KL inequality, the third one uses (52), the fourth Lemma 5.7 and the last inequality uses Lemma 5.10. For convenience, as in [10], we denote  $\Delta_k := \varphi(F(x_k) - \bar{F}) - \varphi(F(x_{k+1}) - \bar{F})$  and we have for any  $N$ ,

$$\sum_{k=k_0}^{N+k_0} \Delta_k \leq \varphi(F(x_{k_0}) - \bar{F}) < +\infty. \quad (59)$$

Then, we can derive from (58) that

$$r_k^3 \leq \frac{6D}{L_H} \Delta_k r_{k-1}. \quad (60)$$

Let  $M := \frac{6D}{L_H}$ . We derive the following inequality by computing 2/3-th order roots on both sides of the above inequality:

$$\begin{aligned} r_k^2 &\leq \left( (M\Delta_k)^2 r_{k-1}^2 \right)^{1/3} \\ &\leq \frac{r_{k-1}^2}{3} + \frac{M\Delta_k}{3} + \frac{M\Delta_k}{3} \\ &\leq \frac{r_{k-1}^2}{3} + \frac{2M\Delta_k}{3} \end{aligned} \quad (61)$$

where the second inequality holds due to the AM-GM inequality  $\frac{a+b+c}{3} \geq (abc)^{1/3}$ . Using Lemma B.1, (61) implies

$$\sum_{k=k_0}^{\infty} r_k^2 \leq \frac{3r_{k_0}^2}{2} + M(\varphi(F(x_{k_0}) - \bar{F})) < +\infty. \quad (62)$$

Using Cauchy-Schwarz inequality, for any  $N$ , we have

$$\sum_{k=k_0}^{N-1+k_0} r_k \leq \left( \sum_{k=k_0}^{N-1+k_0} r_k^2 \right)^{1/2} \left( \sum_{k=k_0}^{N-1+k_0} 1^2 \right)^{1/2} \leq C_0 N^{1/2} \quad (63)$$

where  $C_0 := \sqrt{\frac{3r_{k_0}^2}{2} + M(\varphi(F(x_{k_0}) - \bar{F}))}$ . By definition of  $\lambda_k$ , we can derive that

$$\sum_{k=1+k_0}^{N+k_0} \lambda_k \leq L_H \sum_{k=1+k_0}^{N+k_0} (r_k + r_{k-1}) \leq 2L_H C_0 N^{1/2}. \quad (64)$$

□

The following inequality will be crucial for achieving non-asymptotic convergence rates.

**Lemma 5.12.** *For any  $k \geq k_0$ , we have*

$$\left\langle u_k^\top, (\tilde{G}_{k+1} - J_k) u_k \right\rangle \leq C_1 \|F'(x_k)\|, \quad (65)$$

where  $C_1 := 6\varphi(F(x_{k_0}) - \bar{F})$ .

Furthermore, if  $\varphi(t) = ct^{1-\theta}$  with  $c > 0$  and  $\theta \in (0, 1]$ ,

$$\left\langle u_k^\top, (\tilde{G}_{k+1} - J_k) u_k \right\rangle \leq C_2 \|F'(x_k)\|^{1/\theta}, \quad (66)$$

where  $C_2 := 6((1-\theta)c)^{1/\theta}$ .

*Proof.* Let  $k_0$  be the same one in Lemma 5.11. We start with the first result for general  $\varphi(t)$ .

From the KL inequality and concavity of  $\varphi$ , we have

$$\begin{aligned} 1 &\leq \varphi'(F(x_k) - \bar{F}) \text{dist}(0, \partial F(x_k)) \\ &\leq \frac{\varphi(F(x_k) - \bar{F}) - \varphi(F(x_{k+1}) - \bar{F})}{F(x_k) - F(x_{k+1})} \text{dist}(0, \partial F(x_k)) \\ &\leq \frac{\varphi(F(x_k) - \bar{F})}{F(x_k) - F(x_{k+1})} \|F'(x_k)\|. \end{aligned} \quad (67)$$

By applying (67) along with the monotonicity of  $\varphi$  and  $F(x_k) - \bar{F}$ , we deduce that

$$F(x_k) - F(x_{k+1}) \leq \varphi(F(x_{k_0}) - \bar{F}) \|F'(x_k)\|. \quad (68)$$

Combining (44) and (45), we obtain the first desired inequality:

$$\begin{aligned} \langle (\tilde{G}_{k+1} - J_k)u_k, u_k \rangle &\leq 2(F(x_k) - F(x_{k+1})) + \frac{2L_H}{3} r_k^3, \\ &\leq 6(F(x_k) - F(x_{k+1})) \\ &\leq 6\varphi(F(x_{k_0}) - \bar{F}) \|F'(x_k)\|. \end{aligned} \quad (69)$$

Similarly, we prove the second result. For the Łojasiewicz case where  $\varphi(t) = ct^{1-\theta}$  and  $\theta \in (0, 1)$ , we have

$$\begin{aligned} 1 &\leq \varphi'(F(x_k) - \bar{F}) \text{dist}(0, \partial F(x_k)) \\ &= (1 - \theta)c \frac{1}{(F(x_k) - \bar{F})^\theta} \text{dist}(0, \partial F(x_k)) \\ &\leq ((1 - \theta)c) \frac{1}{(F(x_k) - \bar{F})^\theta} \|F'(x_k)\|. \end{aligned} \quad (70)$$

Thus, we deduce that

$$(F(x_k) - \bar{F}) \leq ((1 - \theta)c)^{1/\theta} \|F'(x_k)\|^{1/\theta}. \quad (71)$$

Using this instead of (68) in the last step of (69), we get (66).  $\square$

**Remark 5.13.** In particular, when  $\varphi(t) = ct^{1/2}$  for some  $c > 0$ , we have for  $k \geq k_0$ ,

$$\langle u_k^\top, (\tilde{G}_{k+1} - J_k)u_k \rangle \leq \frac{3c^2}{2} \|F'(x_k)\|^2. \quad (72)$$

**Lemma 5.14.** For any  $k \in \mathbb{N}$  and  $k \geq k_0$ , we have the inequality:

$$V(\tilde{G}_{k+1}) - V(\tilde{G}_{k+2}) \geq \frac{g_{k+1}^2}{C_1 g_k} - n\lambda_{k+1}, \quad (73)$$

where  $g_k := \|F'(x_k)\|$  and  $C_1 = 6\varphi(F(x_{k_0}) - \bar{F})$ .

Furthermore, if we assume  $\varphi(t) = ct^{1-\theta}$ , then, we have

$$V(\tilde{G}_{k+1}) - V(\tilde{G}_{k+2}) \geq \frac{g_{k+1}^2}{\min\{C_1 g_k, C_2 g_k^{1/\theta}\}} - n\lambda_{k+1}, \quad (74)$$

where  $C_1 := 6\varphi(F(x_{k_0}) - \bar{F})$  and  $C_2 = 6((1 - \theta)c)^{1/\theta}$ .

In particular, if  $\theta = 1/2$ , we have

$$V(\tilde{G}_{k+1}) - V(\tilde{G}_{k+2}) \geq \frac{4g_{k+1}^2}{3c^2 g_k^2} - n\lambda_{k+1}. \quad (75)$$

*Proof.* By the optimality condition in (40), we know that  $\tilde{G}_{k+1}u_k + v_{k+1} = -\nabla f(x_k)$  and by definition of  $J_k$ , we have  $J_k u_k = \nabla f(x_{k+1}) - \nabla f(x_k)$ . Using Lemma 5.5, we obtain:

$$\begin{aligned} V(\tilde{G}_{k+1}) - V(G_{k+1}) &= \nu(J_k, \tilde{G}_{k+1}, u_k) = \frac{\|(\tilde{G}_{k+1} - J_k)u_k\|^2}{u_k^\top (\tilde{G}_{k+1} - J_k)u_k} = \frac{\|\nabla f(x_{k+1}) + v_{k+1}\|^2}{u_k^\top (\tilde{G}_{k+1} - J_k)u_k} \\ &= \frac{\|F'(x_{k+1})\|^2}{u_k^\top (\tilde{G}_{k+1} - J_k)u_k} \geq \frac{\|F'(x_{k+1})\|^2}{C_1 \|F'(x_k)\|} = \frac{g_{k+1}^2}{C_1 g_k}, \end{aligned} \quad (76)$$

where the second inequality holds due to Lemma 5.12. For the case  $\varphi(t) = ct^{1/\theta}$ , we derive from Lemma 5.12 that

$$\begin{aligned} V(\tilde{G}_{k+1}) - V(G_{k+1}) &= \nu(J_k, \tilde{G}_{k+1}, u_k) = \frac{\|F'(x_{k+1})\|^2}{u_k^\top (\tilde{G}_{k+1} - J_k)u_k} \\ &\geq \frac{\|F'(x_{k+1})\|^2}{\min\{C_1 \|F'(x_k)\|, C_2 \|F'(x_k)\|^{1/\theta}\}} \\ &= \frac{g_{k+1}^2}{\min\{C_1 g_k, C_2 g_k^{1/\theta}\}}. \end{aligned} \quad (77)$$



We also need a lower bound for  $V(G_{k+1}) - V(\tilde{G}_{k+2})$ , for which we need to consider the two cases of Step 1 separately.

1. When  $\text{tr } G_{k+1} \leq n\bar{\kappa}$ , we have  $\tilde{G}_{k+2} = G_{k+1} + \lambda_{k+1}I$ , and thus,

$$V(G_{k+1}) - V(\tilde{G}_{k+2}) = \text{tr}(G_{k+1} - G_{k+1} - \lambda_{k+1}I) = -n\lambda_{k+1}. \quad (78)$$

2. When  $\text{tr } G_{k+1} > n\bar{\kappa}$ , we have  $V(G_{k+1}) > n\bar{\kappa} \geq nL = \text{tr } LI$ . In this case we know that  $\tilde{G}_{k+2} = (L + \lambda_{k+1})I$ . Since  $\lambda_k \geq 0$ , we deduce that

$$\begin{aligned} V(G_{k+1}) - V(\tilde{G}_{k+2}) &= V(G_{k+1}) - V(LI) + V(LI) - V(\tilde{G}_{k+2}) \\ &\geq V(LI) - V(\tilde{G}_{k+2}) \\ &\geq \text{tr}(LI - \tilde{G}_{k+2}) \\ &\geq -n\lambda_{k+1}. \end{aligned} \quad (79)$$

Therefore, adding (76) ((77)) with (78) ((79)) yields the desired inequality.  $\square$

Now, we are ready to prove our main theorem for Algorithm 1.

**Proof of Theorem 4.2.** For simplicity, we denote  $\gamma_k^2 := \frac{g_{k+1}^2}{a_k}$ , where  $a_k = C_1 g_k$  for general  $\varphi$  and  $a_k = \min\{C_1 g_k, C_2 g_k^{1/\theta}\}$  for  $\varphi(t) = ct^{1-\theta}$ . Summing the both sides of (73) and (74) from  $k = k_0$  to  $N + k_0 - 1$ , we obtain

$$V(\tilde{G}_{k_0+1}) - V(\tilde{G}_{N+k_0+1}) \geq \sum_{k=k_0}^{N-1+k_0} \gamma_k^2 - n \sum_{k=1+k_0}^{N+k_0} \lambda_k. \quad (80)$$

Since, for every  $k$ , our method keeps  $\tilde{G}_{k+1} \succeq J_k$  and  $G_{k+1} \succeq J_k$  (see Lemma 5.6), we have  $V(\tilde{G}_k) > -nL$  and therefore

$$V(\tilde{G}_{k_0+1}) + nL \geq \sum_{k=k_0}^{N-1+k_0} \gamma_k^2 - n \sum_{k=1+k_0}^{N+k_0} \lambda_k. \quad (81)$$

By Lemma 5.11, we obtain

$$C_{\text{CR1}} + C_{\text{CR2}} N^{1/2} \geq V(\tilde{G}_{k_0+1}) + nL + 2nL_H C_0 N^{1/2} \geq \sum_{k=k_0}^{N-1+k_0} \gamma_k^2, \quad (82)$$

where  $C_{\text{CR1}} := (n+1)L + 2n\bar{\kappa} + 2L_H R \geq nL + V(\tilde{G}_{k_0+1})$  and  $C_{\text{CR2}} := 2nL_H C_0$ . Dividing (82) by  $N$ , we obtain

$$\frac{C_{\text{CR1}}}{N} + \frac{C_{\text{CR2}}}{N^{1/2}} \geq \frac{1}{N} \sum_{k=k_0}^{N-1+k_0} \gamma_k^2. \quad (83)$$

We derive from (83) with the concavity and monotonicity of  $\log x$  that

$$\log \left( \frac{C_{\text{CR1}}}{N} + \frac{C_{\text{CR2}}}{N^{1/2}} \right) \geq \log \left( \frac{1}{N} \sum_{k=k_0}^{N-1+k_0} \gamma_k^2 \right) \geq \frac{1}{N} \sum_{k=k_0}^{N-1+k_0} \log(\gamma_k^2) = \log \left( \left( \prod_{k=k_0}^{N-1+k_0} \frac{g_{k+1}^2}{a_k} \right)^{1/N} \right) \quad (84)$$

We now discuss the two cases of Lemma 5.14 separately, depending on the desingularizing function  $\varphi$ . For general  $\varphi$ , we have  $a_k = C_1 g_k$ . (84) reads

$$\log \left( \frac{C_{\text{CR1}}}{N} + \frac{C_{\text{CR2}}}{N^{1/2}} \right) \geq \log \left( \left( \frac{g_{N+k_0} (\prod_{k=1+k_0}^{N+k_0} g_k)}{C_1^N g_{k_0}} \right)^{1/N} \right). \quad (85)$$

Thus, we obtain

$$\left( \min_{\{k \in \mathbb{N} | k_0 \leq k \leq N+k_0\}} g_k \right)^{\frac{N+1}{N}} \leq \left( g_{N+k_0}^{1/N} \right) \left( \prod_{k=1+k_0}^{N+k_0} g_k \right)^{1/N} \leq C_1 \left( \frac{C_{\text{CR1}}}{N} + \frac{C_{\text{CR2}}}{N^{1/2}} \right) g_{k_0}^{1/N}. \quad (86)$$

Taking the power  $N/(N+1)$  on both sides shows the desired claim in (8):

$$\left( \min_{\{k|k_0 \leq k \leq N+k_0\}} g_k \right) \leq \left( C_1 \left( \frac{C_{\text{CR1}}}{N} + \frac{C_{\text{CR2}}}{N^{1/2}} \right) \right)^{N/(N+1)} g_{k_0}^{2/(N+1)}. \quad (87)$$

For  $\varphi(t) = ct^{1-\theta}$  with  $\theta \in (0, \frac{1}{2}]$ , we have  $a_k = \min\{C_1 g_k, C_2 g_k^{1/\theta}\}$ . Let  $A = C_1 g_k$  and  $B = C_2 g_k^{1/\theta}$ . We notice that by Jensen's inequality

$$\min\{A, B\} \leq A^{\frac{1-2\theta}{1-\theta}} B^{\frac{\theta}{1-\theta}}.$$

Then,  $a_k \leq C g_k^2$  where  $C = C_1^{\frac{1-2\theta}{1-\theta}} C_2^{\frac{\theta}{1-\theta}}$ . In particular, when  $\theta = \frac{1}{2}$ , we have  $C = \frac{3c^2}{4}$ .

By (84), we have

$$\log \left( \frac{C_{\text{CR1}}}{N} + \frac{C_{\text{CR2}}}{N^{1/2}} \right) \geq \log \left( \left( \prod_{k=k_0}^{N-1+k_0} \frac{g_{k+1}^2}{a_k} \right)^{1/N} \right) \geq \log \left( \frac{1}{C} \left( \frac{g_{N+k_0}}{g_{k_0}} \right)^{2/N} \right). \quad (88)$$

Thus, we obtain

$$g_{N+k_0} \leq \left( C \left( \frac{C_{\text{CR1}}}{N} + \frac{C_{\text{CR2}}}{N^{1/2}} \right) \right)^{N/2} g_{k_0}. \quad (89)$$

For  $\varphi(t) = ct^{1-\theta}$  with  $\theta \in (\frac{1}{2}, 1)$ , we have  $a_k \leq \min\{C_1 g_k, C_2 g_k^{1/\theta}\} \leq C_2 g_k^{1/\theta}$ . Thus,

$$\log \left( \frac{C_{\text{CR1}}}{N} + \frac{C_{\text{CR2}}}{N^{1/2}} \right) \geq \log \left( \left( \prod_{k=k_0}^{N-1+k_0} \frac{g_{k+1}^2}{a_k} \right)^{1/N} \right) = \log \left( \left( \frac{1}{C_2} \right) \left( \frac{g_N^2 (\prod_{k=1+k_0}^{N-1+k_0} g_k^{2-\frac{1}{\theta}})}{g_{k_0}^{1/\theta}} \right)^{1/N} \right). \quad (90)$$

Thus, we obtain

$$\min_{\{k|k_0 \leq k \leq N+k_0\}} g_k \leq \left( C_2 \left( \frac{C_{\text{CR1}}}{N} + \frac{C_{\text{CR2}}}{N^{1/2}} \right) \right)^{N/(2+(N-1)(2-\frac{1}{\theta}))} g_{k_0}^{1/(\theta(2+(N-1)(2-\frac{1}{\theta})))}. \quad (91)$$

□

### 5.3 Convergence analysis of Algorithm 2

The necessary optimality condition of the update step in (15) implies

$$\nabla f(x_k) + v_{k+1} + \tilde{G}_k(x_{k+1} - x_k) = 0, \quad (92)$$

where  $v_{k+1} := -\nabla f(x_k) - \tilde{G}_k(x_{k+1} - x_k) \in \partial g(x_{k+1})$ . Thus,  $F'(x_k) \in \partial F(x_k)$ .

**Lemma 5.15.** *For each  $k \in \mathbb{N}$ , we have*

$$J_k \preceq G_{k+1} \preceq \tilde{G}_k, \quad (93)$$

$$V(G_{k+1}) \leq V(\tilde{G}_k) \leq n\bar{\kappa}. \quad (94)$$

*Proof.* We first notice that since  $g$  is convex, the monotonicity of its subdifferential yields

$$\langle u_k, v_{k+1} - v_k \rangle \geq 0, \quad (95)$$

for any  $k \in \mathbb{N}$ . Thus, for any  $k \in \mathbb{N}$ , we have

$$\langle u_k, \tilde{G}_k u_k \rangle \leq \langle u_k, \tilde{G}_k u_k \rangle + \langle u_k, v_{k+1} - v_k \rangle = \langle u_k, -\nabla f(x_k) - v_{k+1} + v_{k+1} - v_k \rangle = -\langle u_k, F'(x_k) \rangle. \quad (96)$$

The first inequality in (96) is a direct consequence of (95).

We prove the remaining inequalities in (93) by induction. In order to validate the base case  $k = 0$ , we first observe that  $\tilde{G}_0 = G_0 + \lambda_0 I \succeq J_0$  holds by Lipschitz continuity of  $\nabla f$  and  $G_0 = LI$  and  $\lambda_0 = 0$ . Then, Lemma 5.4 shows the desired property:

$$J_0 \preceq G_1 = \text{SR1}(J_0, \tilde{G}_0, u_0) \preceq \tilde{G}_0 \preceq nLI \preceq n\bar{\kappa}I.$$

For showing the induction, we suppose now that (93) holds for  $k - 1$ . We discuss each case separately.

1. If  $\text{tr } \hat{G}_k \leq n\bar{\kappa}$ , then,  $\tilde{G}_k = \hat{G}_k$ . Since  $\tilde{G}_k = G_k + \lambda_k I \succeq \lambda_k I$ , the above equality (96) implies

$$\lambda_k \|u_k\|^2 \leq u_k^\top (G_k + \lambda_k I) u_k = u_k^\top \tilde{G}_k u_k \leq -u_k^\top F'(x_k) \leq \|u_k\| \|F'(x_k)\|. \quad (97)$$

Thus,  $\|u_k\| \leq \frac{\|F'(x_k)\|}{\lambda_k}$ . Therefore,

$$L_H r_k = L_H \|u_k\| \leq \frac{L_H \|F'(x_k)\|}{\lambda_k} \leq \sqrt{L_H \|F'(x_k)\|}, \quad (98)$$

where the last inequality holds since  $\lambda_k \geq \sqrt{L_H \|F'(x_k)\|}$  (see the definition of  $\lambda_k$ ). We can deduce that:

$$L_H r_k + L_H r_{k-1} \leq \sqrt{L_H \|F'(x_k)\|} + L_H r_{k-1} = \lambda_k. \quad (99)$$

Therefore, we deduce by the induction hypothesis and Lemma 5.3 that

$$\begin{aligned} \tilde{G}_k &= G_k + \lambda_k I \\ &\succeq J_{k-1} + \lambda_k I \\ &\succeq J_{k-1} + (L_H r_{k-1} + L_H r_k) I \\ &\succeq J_k. \end{aligned} \quad (100)$$

We are then in position to apply Lemma 5.4 again to get  $J_k \preceq G_{k+1} = \text{SR1}(J_k, \tilde{G}_k, u_k) \preceq \tilde{G}_k$  and  $\text{tr } G_{k+1} \leq \text{tr } \tilde{G}_k \leq n\bar{\kappa}$ .

2. If  $\text{tr } \hat{G}_N > n\bar{\kappa}$ , then  $\tilde{G}_N = LI$  due to the restarting step. Automatically, we have  $\tilde{G}_N \succeq J_N$  and using Lemma 5.4 again, we obtain  $J_N \preceq G_{N+1} \preceq \tilde{G}_N \preceq LI$  and  $\text{tr } G_{N+1} \leq \text{tr } \tilde{G}_N = nL \leq n\bar{\kappa}$ .  $\square$

**Lemma 5.16.** *For any  $k \in \mathbb{N}$  and  $t \in [0, 1]$ , we have*

$$\nabla^2 f(x_k + tu_k) \preceq \tilde{G}_k. \quad (101)$$

*Proof.* Following the proof for Lemma 5.3, we have

$$\nabla^2 f(x_k + tu_k) \preceq J_{k-1} + \frac{1}{2} L_H r_{k-1} + \frac{1}{2} L_H r_k. \quad (102)$$

By Lemma 5.15, we have  $J_{k-1} \preceq G_k$ . Thus, for the case  $\hat{G}_k \leq n\bar{\kappa}$ , we have

$$\nabla^2 f(x_k + tu_k) \preceq J_{k-1} + \frac{1}{2} L_H r_{k-1} + \frac{1}{2} L_H r_k \preceq G_k + \frac{1}{2} L_H r_{k-1} + \frac{1}{2} L_H r_k = \tilde{G}_k. \quad (103)$$

Otherwise,  $\tilde{G}_k = LI$ . We still have

$$\nabla^2 f(x_k + tu_k) \preceq \tilde{G}_k. \quad (104)$$

$\square$

**Lemma 5.17.** *For any  $k \in \mathbb{N}$ , we have*

$$F(x_{k+1}) - F(x_k) \leq -\frac{1}{2} \langle \tilde{G}_k u_k, u_k \rangle. \quad (105)$$

*Proof.* We start with the Taylor expansion of the smooth function  $f$  and we obtain

$$\begin{aligned} f(x_{k+1}) - f(x_k) &= \langle \nabla f(x_k), u_k \rangle + \int_0^1 (1-t) \langle \nabla^2 f(x_k + tu_k) u_k, u_k \rangle dt \\ &\leq \langle \nabla f(x_k), u_k \rangle + \int_0^1 (1-t) \langle \tilde{G}_k u_k, u_k \rangle dt \\ &= \langle \nabla f(x_k), u_k \rangle + \frac{1}{2} \langle \tilde{G}_k u_k, u_k \rangle, \end{aligned} \quad (106)$$

where the first inequality holds due to Lemma 5.16. Since  $x_{k+1}$  is the solution of the inclusion (92) and  $g$  is convex, the subgradient inequality holds:

$$g(x_{k+1}) - g(x_k) \leq \langle v_{k+1}, u_k \rangle. \quad (107)$$

Combining (106) and (107), we deduce that

$$\begin{aligned} F(x_{k+1}) - F(x_k) &\leq \langle v_{k+1} + \nabla f(x_k), u_k \rangle + \frac{1}{2} \langle \tilde{G}_k u_k, u_k \rangle \\ &= -\frac{1}{2} \langle \tilde{G}_k u_k, u_k \rangle, \end{aligned} \quad (108)$$

where we used (92) to get the last equality.  $\square$

Due to the monotone decrease of the function values, we have for any  $k$ ,  $x_k \in [F \leq F(x_0)]$ .

**Lemma 5.18.** *Assume  $(x_k)_{k \in \mathbb{N}}$  generated by Algorithm 2 is bounded. We have the following results:*

1.  $\|F'(x_k)\| \rightarrow 0$  and  $F(x_k) \rightarrow \min F$ ,
2.  $\omega(x_0) \subset \text{crit} F$ ,
3. The objective function  $F(x) \equiv \min F$  over  $\Omega$  where  $\min F$  is finite.

*Proof.* The argument remains the same as that of Lemma 5.9.  $\square$

### 5.3.1 Proof of Theorem 4.9

**Lemma 5.19.** *Given a number of iteration  $N \in \mathbb{N}$  and  $k \geq k_0$  for large enough  $k_0$ , we have:*

$$\|F'(x_k)\| \leq \frac{\lambda_k^2}{L_H}, \quad (109)$$

$$\lambda_{k+1} \leq D\sqrt{r_k}, \quad (110)$$

$$\sum_{k=1+k_0}^{N+k_0} \lambda_k \leq DC_0^{1/3} N^{2/3}, \quad (111)$$

where  $C_0 := \frac{r_{k_0}^{3/2}}{3} + M(\varphi(F(x_{k_0}) - \inf F))$ ,  $M = \sqrt{\frac{4(n\bar{\kappa}+L)}{L_H}}$ ,  $D := (\sqrt{L_H(n\bar{\kappa}+L)} + L_H R)$  and  $R \geq \max_{k \in \mathbb{N}} r_k$ .

*Proof.* Thanks to Assumption 1, 2 and Lemma 3.1,  $F$  has uniformized KL property with respect to  $\omega(x_0)$ . We recall that we assume  $\text{argmin } F \neq \emptyset$ . Then, there exist  $\epsilon > 0$ ,  $\eta > 0$  and  $\varphi \in \Phi_\eta$  such that for all  $\bar{x} \in \omega(x_0)$  and all  $x$  in the following intersection

$$\{x \in \mathbb{R}^n | \text{dist}(x, \Omega) < \epsilon\} \cap [F(\bar{x}) < F < F(\bar{x}) + \eta], \quad (112)$$

one has,

$$\varphi'(F(x) - F(\bar{x})) \text{dist}(0, \partial F(x)) \geq 1. \quad (113)$$

Thanks to Lemma 5.17 and 5.18, there exists some  $k_0$  such that for any  $k \geq k_0$ ,

$$x_k \in \{x \in \mathbb{R}^n | \text{dist}(x, \Omega) < \epsilon\} \cap [\min F < F(x) < \min F + \eta].$$

We start with the first inequality. Step 3 implies that  $\lambda_k \geq \sqrt{L_H \|F'(x_k)\|}$  since  $r_k \geq 0$  for any  $k \in \mathbb{N}$ . Thus, we obtain

$$\|F'(x_k)\| \leq \frac{\lambda_k^2}{L_H}. \quad (114)$$

Now, we prove the second inequality. From the definition of  $F'(x_{k+1})$ , it follows that

$$\begin{aligned} \|F'(x_{k+1})\| &= \|-\tilde{G}_k u_k - (\nabla f(x_k) - \nabla f(x_{k+1}))\| \\ &\leq \|\tilde{G}_k u_k\| + \|(\nabla f(x_k) - \nabla f(x_{k+1}))\| \\ &\leq n\bar{\kappa} r_k + L r_k \\ &\leq (n\bar{\kappa} + L) r_k. \end{aligned} \quad (115)$$

Since the boundedness of  $(x_k)_{k \in \mathbb{N}}$  is assumed, there exists some  $R > 0$  such that for any  $k \in \mathbb{N}$ ,  $r_k \leq R$ . According to step 3, we have

$$\begin{aligned} \lambda_{k+1} &= \sqrt{L_H \|F'(x_{k+1})\|} + L_H r_k \\ &\leq \sqrt{L_H (n\bar{\kappa} + L) r_k} + L_H r_k \\ &\leq D\sqrt{r_k}, \end{aligned} \quad (116)$$

where  $D = (\sqrt{L_H(n\bar{\kappa} + L)} + L_H\sqrt{R})$ .

Finally, we are going to show the third inequality. According to Lemma 5.17, we have

$$F(x_{k+1}) - F(x_k) \leq -\frac{1}{2} \langle \tilde{G}_k u_k, u_k \rangle \leq -\frac{1}{2} \lambda_k r_k^2, \quad (117)$$

where the last inequality holds since  $G_k$  is symmetric positive semi-definite and  $\tilde{G}_k = G_k + \lambda_k I$ .

Since  $F$  satisfies KL inequality, we adopt the proof from [10]. Using (114), (115) and (117), we obtain: for any  $k \geq k_0$ ,

$$\begin{aligned} \varphi(F(x_k) - \inf F) - \varphi(F(x_{k+1}) - \inf F) &\geq \varphi'(F(x_k) - \inf F)(F(x_k) - F(x_{k+1})) \\ &\geq \frac{(F(x_k) - F(x_{k+1}))}{\text{dist}(0, \partial F(x_k))} \geq \frac{(F(x_k) - F(x_{k+1}))}{\|F'(x_k)\|} \\ &\geq \frac{\lambda_k r_k^2}{2\|F'(x_k)\|} \\ &\geq \frac{\lambda_k r_k^2}{2\|F'(x_k)\|^{1/2} \|F'(x_k)\|^{1/2}} \\ &\geq \frac{\lambda_k r_k^2}{2\sqrt{\frac{\lambda_k^2}{L_H} \sqrt{(n\bar{\kappa} + L)} r_{k-1}}} \\ &\geq \frac{\sqrt{L_H} r_k^2}{2\sqrt{(n\bar{\kappa} + L)} r_{k-1}} \end{aligned} \quad (118)$$

where the first inequality uses the concavity of  $\varphi$ . For convenience, as in [10], we denote  $\Delta_k := \varphi(F(x_k) - \inf F) - \varphi(F(x_{k+1}) - \inf F)$  and we have for any  $N \in \mathbb{N}$ ,

$$\sum_{k=k_0}^{N+k_0} \Delta_k \leq \varphi(F(x_{k_0}) - \inf F) < +\infty. \quad (119)$$

Then, we can derive from (118) that

$$r_k^2 \leq \sqrt{\frac{4(n\bar{\kappa} + L)}{L_H}} \Delta_k \sqrt{r_{k-1}}. \quad (120)$$

Let  $M$  denote  $\sqrt{\frac{4(n\bar{\kappa} + L)}{L_H}}$ . We derive the following inequality by computing 3/4-th order root on both sides of the above inequality:

$$\begin{aligned} r_k^{3/2} &\leq \left( (M\Delta_k)^3 r_{k-1}^{3/2} \right)^{1/4} \\ &\leq \frac{r_{k-1}^{3/2}}{4} + \frac{M\Delta_k}{4} + \frac{M\Delta_k}{4} + \frac{M\Delta_k}{4} \\ &\leq \frac{r_{k-1}^{3/2}}{4} + \frac{3M\Delta_k}{4} \end{aligned} \quad (121)$$

where the second inequality holds due to the AM-GM inequality  $\frac{a+b+c+d}{4} \geq (abcd)^{1/4}$ . Summing up (121) from  $1 + k_0$  to any  $N + k_0$ , by reorganizing, we have

$$\sum_{k=1+k_0}^{N+k_0} r_k^{3/2} \leq \frac{r_{k_0}^{3/2}}{3} + M \sum_{k=1+k_0}^{N+k_0} \Delta_k \leq \frac{r_{k_0}^{3/2}}{3} + M(\varphi(F(x_{k_0}) - \inf F)) < +\infty. \quad (122)$$

We denote  $C_0 = \frac{r_{k_0}^{3/2}}{3} + M(\varphi(F(x_{k_0}) - \inf F))$ . Using Hölder inequality, for any  $N$ , we have

$$\sum_{k=1+k_0}^{N+k_0} r_k^{1/2} \leq \left( \sum_{k=1+k_0}^{N+k_0} r_k^{3/2} \right)^{1/3} \left( \sum_{k=1+k_0}^{N+k_0} 1^{3/2} \right)^{2/3} \leq C_0^{1/3} N^{2/3} \quad (123)$$

$$\sum_{k=1+k_0}^{N+k_0} r_k \leq \left( \sum_{k=1+k_0}^{N+k_0} r_k^{3/2} \right)^{2/3} \left( \sum_{k=1+k_0}^{N+k_0} 1^3 \right)^{1/3} \leq C_0^{2/3} N^{1/3}. \quad (124)$$

From (116), we can derive that

$$\sum_{k=1+k_0}^{N+k_0} \lambda_k \leq D \sum_{k=1+k_0}^{N+k_0} r_k^{1/2} \leq DC_0^{1/3} N^{2/3}. \quad (125)$$

□

**Lemma 5.20.** *For any  $k \geq k_0$  where  $k_0$  is large enough, we have*

$$\langle u_k, \tilde{G}_k u_k \rangle \leq S_1 \|F'(x_k)\|, \quad (126)$$

where  $S_1 := 2\varphi(F(x_{k_0}) - \inf F)$ . Furthermore, if  $\varphi(t) = ct^{1-\theta}$  with  $c > 0$ ,

$$\langle u_k, \tilde{G}_k u_k \rangle \leq \min\{S_2 \|F'(x_k)\|^{1/\theta}, S_1 \|F'(x_k)\|\}, \quad (127)$$

where  $S_2 = 2((1-\theta)c)^{1/\theta}$ .

*Proof.* Similar to the proof of Lemma 5.19, there exists  $\epsilon > 0$ ,  $\eta > 0$  and  $\varphi \in \Phi_\eta$  such that for all  $\bar{x} \in \omega(x_0)$  and all  $x$  in the following intersection

$$\{x \in \mathbb{R}^n | \text{dist}(x, \Omega) < \epsilon\} \cap [F(\bar{x}) < F < F(\bar{x}) + \eta], \quad (128)$$

one has,

$$\varphi'(F(x) - F(\bar{x})) \text{dist}(0, \partial F(x)) \geq 1. \quad (129)$$

Then, there exists some  $k_0$  such that for any  $k \geq k_0$ ,

$$x_k \in \{x \in \mathbb{R}^n | \text{dist}(x, \Omega) < \epsilon\} \cap [\min F < F(x) < \min F + \eta].$$

We start with the first result for general  $\varphi(t)$ . From the KL inequality and concavity of  $\varphi$ , we have for any  $k \geq k_0$ ,

$$\begin{aligned} 1 &\leq \varphi'(F(x_k) - \inf F) \text{dist}(0, \partial F(x_k)) \\ &\leq \frac{\varphi(F(x_k) - \inf F) - \varphi(F(x_{k+1}) - \inf F)}{F(x_k) - F(x_{k+1})} \text{dist}(0, \partial F(x_k)) \\ &\leq \frac{\varphi(F(x_k) - \inf F)}{F(x_k) - F(x_{k+1})} \|F'(x_k)\|. \end{aligned} \quad (130)$$

From (130), using Lemma 5.17, we deduce that

$$(F(x_k) - F(x_{k+1})) \leq \varphi(F(x_k) - \inf F) \|F'(x_k)\| \leq \varphi(F(x_{k_0}) - \inf F) \|F'(x_k)\|, \quad (131)$$

where the first inequality holds since  $\varphi$  is increasing and the sequence  $F(x_k)$  is nonincreasing. According to Lemma 5.17, we have

$$u_k^\top \tilde{G}_k u_k \leq 2(F(x_k) - F(x_{k+1})) \leq 2\varphi(F(x_{k_0}) - \inf F) \|F'(x_k)\|. \quad (132)$$

Similarly, we prove the second result. For the Łojasiewicz case where  $\varphi(t) = ct^{1-\theta}$ , we have

$$\begin{aligned} 1 &\leq \varphi'(F(x_k) - \inf F) \text{dist}(0, \partial F(x_k)) \\ &= (1-\theta)c \frac{1}{(F(x_k) - \inf F)^\theta} \text{dist}(0, \partial F(x_k)) \\ &\leq ((1-\theta)c)^{1/\theta} \frac{1}{(F(x_{k_0}) - \inf F)^\theta} \|F'(x_k)\|. \end{aligned} \quad (133)$$

According to Lemma 5.17, we have

$$u_k^\top \tilde{G}_k u_k \leq 2(F(x_k) - F(x_{k+1})) \leq 2(F(x_{k_0}) - \inf F) \leq 2((1-\theta)c)^{1/\theta} \|F'(x_k)\|^{1/\theta}. \quad (134)$$

By combining (130) and (134), we obtain (127). □

**Remark 5.21.** In particular, when  $\varphi(t) = ct^{1/2}$  for some  $c > 0$ , we have for any  $k \geq k_0$ ,

$$u_k^\top \tilde{G}_k u_k \leq \frac{c^2}{2} \|F'(x_k)\|^2. \quad (135)$$

**Lemma 5.22.** For any  $k \in \mathbb{N}$  and  $k > k_0$ , we have a descent inequality as the following:

$$V(\tilde{G}_k) - V(\tilde{G}_{k+1}) \geq \frac{g_{k+1}^2}{S_1 g_k} - n\lambda_{k+1}, \quad (136)$$

where  $g_k := \|F'(x_k)\|$  and  $S_1 = 2\varphi(F(x_{k_0}) - \inf F)$ . Furthermore, if we assume  $\varphi(t) = ct^{1-\theta}$ , then, we have

$$V(\tilde{G}_k) - V(\tilde{G}_{k+1}) \geq \frac{g_{k+1}^2}{\min\{S_1 g_k, S_2 g_k^{1/\theta}\}} - n\lambda_{k+1}, \quad (137)$$

where  $S_2 = 2((1-\theta)c)^{1/\theta}$ . In particular, if  $\theta = 1/2$ , we have

$$V(\tilde{G}_k) - V(\tilde{G}_{k+1}) \geq \frac{2g_{k+1}^2}{c^2 g_k^2} - n\lambda_{k+1}. \quad (138)$$

*Proof.* By the optimality condition in (92), we know that  $\tilde{G}_k u_k + v_{k+1} = -\nabla f(x_k)$  and by definition of  $J_k$ , we have  $J_k u_k = \nabla f(x_{k+1}) - \nabla f(x_k)$ . Using Lemma 5.5, we obtain:

$$\begin{aligned} V(\tilde{G}_k) - V(G_{k+1}) &= \nu(J_k, \tilde{G}_k, u_k) \geq \frac{\|(\tilde{G}_k - J_k)u_k\|^2}{u_k^\top \tilde{G}_k u_k} = \frac{\|\nabla f(x_{k+1}) + v_{k+1}\|^2}{u_k^\top \tilde{G}_k u_k} \\ &= \frac{\|F'(x_{k+1})\|^2}{u_k^\top \tilde{G}_k u_k} \geq \frac{\|F'(x_{k+1})\|^2}{S_1 \|F'(x_k)\|} = \frac{g_{k+1}^2}{S_1 g_k}, \end{aligned} \quad (139)$$

where the second inequality holds due to Lemma 5.20. For the case  $\varphi(t) = ct^{1/\theta}$ , we derive from Lemma 5.20 that

$$\begin{aligned} V(\tilde{G}_k) - V(G_{k+1}) &= \nu(J_k, \tilde{G}_k, u_k) = \frac{\|F'(x_{k+1})\|^2}{u_k^\top \tilde{G}_k u_k} \\ &\geq \frac{\|F'(x_{k+1})\|^2}{\min\{S_1 \|F'(x_k)\|, S_2 \|F'(x_k)\|^{1/\theta}\}} \\ &= \frac{g_{k+1}^2}{\min\{S_1 g_k, S_2 g_k^{1/\theta}\}}, \end{aligned} \quad (140)$$

We also need a lower bound for  $V(G_{k+1}) - V(\tilde{G}_{k+1})$ , for which we need to consider the two cases of the restarting Step 4.

1. When  $\|\hat{G}_{k+1}\|_2 \leq n\bar{\kappa}$ , we have  $\tilde{G}_{k+1} = \hat{G}_{k+1}$ , and

$$V(G_{k+1}) - V(\tilde{G}_{k+1}) = \text{tr}(G_{k+1} - \tilde{G}_{k+1} - \lambda_{k+1}) = -n\lambda_{k+1}. \quad (141)$$

2. When  $\text{tr} \hat{G}_{k+1} > n\bar{\kappa}$ , we have  $V(\hat{G}_{k+1}) > n\bar{\kappa} \geq nL = \text{tr} LI$ . Since in this case we set  $\tilde{G}_{k+1} = LI$ , we have  $V(\hat{G}_{k+1}) \geq V(\tilde{G}_{k+1})$ . According to Lemma 5.15, we have  $\text{tr} G_{k+1} \leq n\bar{\kappa}$ . Since  $\lambda_k \geq 0$ , we deduce that

$$\begin{aligned} V(G_{k+1}) - V(\tilde{G}_{k+1}) &= V(G_{k+1}) - V(\hat{G}_{k+1}) + V(\hat{G}_{k+1}) - V(\tilde{G}_{k+1}) \\ &\geq V(G_{k+1}) - V(\hat{G}_{k+1}) \\ &\geq \text{tr}(G_{k+1} - \hat{G}_{k+1} - \lambda_{k+1}) \\ &\geq -n\lambda_{k+1} \\ &\geq -n\lambda_{k+1}. \end{aligned} \quad (142)$$

Therefore, adding (139) with either (141) or (142) yields the desired inequality.  $\square$



Now, we are ready to prove our main theorem for Algorithm 2.

**Proof of Theorem 4.9.** For simplicity, we denote  $\gamma_k^2 := \frac{g_{k+1}^2}{a_k}$ , where  $a_k = S_1 g_k$  for general  $\varphi$  and  $a_k = \min\{S_1 g_k, S_2 g_k^{1/\theta}\}$  for  $\varphi(t) = ct^{1-\theta}$ . Summing both sides of (136) and (137) from  $k = k_0$  to  $N - 1 + k_0$ , we obtain

$$V(\tilde{G}_{k_0}) - V(\tilde{G}_{N+k_0}) \geq \sum_{k=k_0}^{N-1+k_0} \gamma_k^2 - n \sum_{k=1+k_0}^{N+k_0} \lambda_k. \quad (143)$$

Since, for every  $k$ , our method keeps  $\tilde{G}_k \succeq J_k$  and  $G_{k+1} \succeq J_k$  (see Lemma 5.15), we have  $V(\tilde{G}_k) > 0$  and therefore

$$V(\tilde{G}_{k_0}) \geq \sum_{k=k_0}^{N-1+k_0} \gamma_k^2 - n \sum_{k=1+k_0}^{N+k_0} \lambda_k. \quad (144)$$

By Lemma 5.19, we obtain

$$V(\tilde{G}_{k_0}) + C_{\text{GR}} N^{2/3} \geq V(\tilde{G}_{k_0}) + n D C_0^{1/3} N^{2/3} \geq \sum_{k=k_0}^{N-1+k_0} \gamma_k^2, \quad (145)$$

where  $C_{\text{GR}} := n D C_0^{1/3}$ . Dividing (145) by  $N$ , we obtain

$$\frac{V(\tilde{G}_{k_0})}{N} + \frac{C_{\text{GR}}}{N^{1/3}} \geq \frac{1}{N} \sum_{k=k_0}^{N-1+k_0} \gamma_k^2. \quad (146)$$

We derive from (146) with the concavity and monotonicity of  $\log x$  that

$$\log \left( \frac{V(\tilde{G}_{k_0})}{N} + \frac{C_{\text{GR}}}{N^{1/3}} \right) \geq \log \left( \frac{1}{N} \sum_{k=k_0}^{N-1+k_0} \gamma_k^2 \right) \geq \frac{1}{N} \sum_{k=k_0}^{N-1+k_0} \log(\gamma_k^2) = \log \left( \left( \prod_{k=k_0}^{N-1+k_0} \frac{g_{k+1}^2}{a_k} \right)^{1/N} \right) \quad (147)$$

We now discuss the two cases of Lemma 5.22 separately, depending on the desingularizing function  $\varphi$ . For general  $\varphi$ , we have  $a_k = S_1 g_k$ . Thus, (147) reads

$$\log \left( \frac{V(\tilde{G}_{k_0})}{N} + \frac{C_{\text{GR}}}{N^{1/3}} \right) \geq \log \left( \left( \frac{g_N (\prod_{k=1+k_0}^{N+k_0} g_k)}{S_1^N g_0} \right)^{1/N} \right). \quad (148)$$

Therefore, we obtain

$$\left( \min_{\{k \in \mathbb{N} | k_0 \leq k \leq N+k_0\}} g_k \right)^{\frac{N+1}{N}} \leq \left( g_N^{1/N} \right) \left( \prod_{k=1}^N g_k \right)^{1/N} \leq \left( S_1 \left( \frac{V(\tilde{G}_{k_0})}{N} + \frac{C_{\text{GR}}}{N^{1/3}} \right) \right) g_{k_0}^{1/N}. \quad (149)$$

Taking the power  $N/(N+1)$  on both sides shows the desired claim in (16).

$$\left( \min_{\{k \in \mathbb{N} | k_0 \leq k \leq N+k_0\}} g_k \right) \leq \left( S_1 \left( \frac{V(\tilde{G}_{k_0})}{N} + \frac{C_{\text{GR}}}{N^{1/3}} \right) \right)^{N/(N+1)} g_{k_0}^{1/(N+1)}. \quad (150)$$

For  $\varphi(t) = ct^{1/2}$ , we have  $a_k = \frac{c^2}{2} g_k^2$ .

$$g_N \leq \left( \frac{c^2}{2} \left( \frac{V(\tilde{G}_{k_0})}{N} + \frac{C_{\text{GR}}}{N^{1/3}} \right) \right)^{N/2} g_{k_0}. \quad (151)$$

For  $\varphi(t) = ct^{1-\theta}$  with  $\theta \in (0, \frac{1}{2})$ , we have  $a_k = \min\{S_1 g_k, S_2 g_k^{1/\theta}\}$ . Let  $A = S_1 g_k$  and  $B = S_2 g_k^{1/\theta}$ . We notice that by Jensen's inequality.

$$\min\{A, B\} \leq A^{\frac{1-2\theta}{1-\theta}} B^{\frac{\theta}{1-\theta}}.$$

Then,  $a_k \leq Sg_k^2$  where  $S = S_1^{\frac{1-2\theta}{1-\theta}} S_2^{\frac{\theta}{1-\theta}}$ .

By (147), we have

$$\log \left( \frac{V(\tilde{G}_{k_0})}{N} + \frac{C_{\text{GR}}}{N^{1/3}} \right) \geq \log \left( \left( \prod_{k=k_0}^{N-1+k_0} \frac{g_{k+1}^2}{a_k} \right)^{1/N} \right) \geq \log \left( \frac{1}{S} \left( \frac{g_N}{g_{k_0}} \right)^{2/N} \right). \quad (152)$$

Therefore, we obtain

$$g_N \leq \left( S \left( \frac{V(\tilde{G}_{k_0})}{N} + \frac{C_{\text{GR}}}{N^{1/3}} \right) \right)^{N/2} g_{k_0} \quad (153)$$

For  $\varphi(t) = ct^{1-\theta}$  with  $\theta \in (\frac{1}{2}, 1)$ , we have  $a_k \leq \min\{S_1 g_k, S_2 g_k^{1/\theta}\} \leq S_2 g_k^{1/\theta}$ . Thus,

$$\log \left( \frac{V(\tilde{G}_{k_0})}{N} + \frac{C_{\text{GR}}}{N^{1/3}} \right) \geq \log \left( \left( \prod_{k=k_0}^{N-1+k_0} \frac{g_{k+1}^2}{a_k} \right)^{1/N} \right) = \log \left( \left( \frac{1}{S_2} \right) \left( \frac{g_N^2 (\prod_{k=1+k_0}^{N-1+k_0} g_k^{2-\frac{1}{\theta}})}{g_{k_0}^{1/\theta}} \right)^{1/N} \right). \quad (154)$$

Thus, we obtain

$$\min_{\{k \in \mathbb{N} | k_0 \leq k \leq N+k_0\}} g_k \leq \left( S_2 \left( \frac{V(\tilde{G}_{k_0})}{N} + \frac{C_{\text{GR}}}{N^{1/3}} \right) g_{k_0}^{1/(\theta N)} \right)^{1/(\frac{2}{N} + (N-1)(2-\frac{1}{\theta}))}. \quad (155) \quad \square$$

### 5.3.2 Proof of Theorem 4.13

When  $g = 0$ , by Lemma 5.17, we can derive the following bound on  $\|\nabla f(x_k)\|$  for any  $k \in \mathbb{N}$ .

**Lemma 5.23.** *For any  $k \in \mathbb{N}$ , we have*

$$\|\nabla f(x_k)\|^2 \leq 2n\bar{\kappa} (f(x_k) - f(x_{k+1})). \quad (156)$$

*Proof.* Thanks to Lemma 5.15, we have

$$\langle \tilde{G}_k u_k, u_k \rangle = \langle \tilde{G}_k u_k, \tilde{G}_k^{-1} \tilde{G}_k u_k \rangle \geq \frac{1}{n\bar{\kappa}} \langle \tilde{G}_k u_k, \tilde{G}_k u_k \rangle = \frac{1}{n\bar{\kappa}} \|\nabla f(x_k)\|^2.$$

Together with Lemma 5.17, we obtain the desired result.  $\square$

**Lemma 5.24.** *Assume  $f$  satisfies (20). For any  $N \in \mathbb{N}$ , we have*

$$f(x_N) - \min f \leq q^N (f(x_0) - \min f) \quad (157)$$

$$\sum_{k=0}^N \|\nabla f(x_k)\|^{1/2} \leq C_f, \quad (158)$$

$$\sum_{k=0}^N r_k \leq C_r \quad (159)$$

where  $q = \frac{cn\bar{\kappa}-1}{cn\bar{\kappa}}$ ,  $C_f = 4n\bar{\kappa}(cn\bar{\kappa}(f(x_0) - \min f))^{1/4}$  and

$$C_r = \frac{1}{\sqrt{L_H}} C_f + \frac{1}{L} \bar{C}_f.$$

*Proof.* By Lemma 5.23 and (20), we have that

$$f(x_k) - \inf f \leq cn\bar{\kappa} (f(x_k) - f(x_{k+1})) \leq cn\bar{\kappa} [f(x_k) - \min f - (f(x_{k+1}) - \min f)]. \quad (160)$$

Here,  $cn\bar{\kappa} > 1$ . Otherwise,  $f(x_{k+1}) \leq \min f$  and we get a contradiction. Then, we deduce that

$$f(x_{k+1}) - \min f \leq q(f(x_k) - \min f), \quad (161)$$

where  $q = \frac{cn\bar{\kappa}-1}{cn\bar{\kappa}}$ . Thus,  $f(x_N) - \min f \leq q^N (f(x_0) - \min f)$ . By Lemma 5.23, we derive from (161) that

$$\|\nabla f(x_N)\|^{1/2} \leq (2n\bar{\kappa}(f(x_0) - \min f))^{1/4} q^{N/4}. \quad (162)$$

Summing (162) up to  $N$ , we have

$$\sum_0^N \|\nabla f(x_N)\|^{1/2} \leq \frac{1}{1-q^{1/4}} (2n\bar{\kappa}(f(x_0) - \min f))^{1/4}, \quad (163)$$

$$\sum_0^N \|\nabla f(x_N)\| \leq \frac{1}{1-q^{1/2}} (2n\bar{\kappa}(f(x_0) - \min f))^{1/2}. \quad (164)$$

Since  $\frac{1}{1-q^{1/4}} \leq \frac{4}{1-q} = 4cn\bar{\kappa}$ , we set  $C_f = 4cn\bar{\kappa}(2n\bar{\kappa}(f(x_0) - \min f))^{1/4}$ . Similarly, we set  $\bar{C}_f = 2cn\bar{\kappa}(2n\bar{\kappa}(f(x_0) - \min f))^{1/2}$ . Thanks to the restarting strategy, we have  $\tilde{G}_N = L$  or  $\tilde{G}_N = G_N + \lambda_N I$  and  $G_N \succeq 0$  for any  $N \in \mathbb{N}$ . Then, we deduce that  $\tilde{G}_N \succeq \min\{L, \lambda_N\}I$ . Thus,

$$\min\{L, \lambda_N\}\|u_N\| \leq \|\tilde{G}_N u_N\| \leq \|\nabla f(x_N)\|. \quad (165)$$

Since  $\lambda_N > \sqrt{L_H \|\nabla f(x_N)\|}$ , (165) implies that

$$\begin{aligned} r_N &= \|u_N\| \\ &\leq \max\left\{\frac{1}{L}\|\nabla f(x_N)\|, \frac{1}{\lambda_N}\|\nabla f(x_N)\|\right\} \\ &\leq \max\left\{\frac{1}{L}\|\nabla f(x_N)\|, \frac{1}{\sqrt{L_H}}\sqrt{\|\nabla f(x_N)\|}\right\} \\ &\leq \frac{1}{L}\|\nabla f(x_N)\| + \frac{1}{\sqrt{L_H}}\sqrt{\|\nabla f(x_N)\|}. \end{aligned} \quad (166)$$

Summing up (166) from 0 to  $N$ , we have

$$\sum_0^N r_k \leq \frac{1}{\sqrt{L_H}} \sum_0^N \|\nabla f(x_N)\|^{1/2} + \frac{1}{L} \sum_0^N \|\nabla f(x_N)\| \leq \frac{1}{\sqrt{L_H}} C_f + \frac{1}{L} \bar{C}_f. \quad (167)$$

We set

$$\begin{aligned} C_r &:= \frac{1}{\sqrt{L_H}} 4cn\bar{\kappa}(2n\bar{\kappa}(f(x_0) - \min f))^{1/4} + \frac{1}{L} 2cn\bar{\kappa}(2n\bar{\kappa}(f(x_0) - \min f))^{1/2} \\ &= \frac{1}{\sqrt{L_H}} C_f + \frac{1}{L} \bar{C}_f. \end{aligned} \quad (168)$$

This shows the desired results.  $\square$

**Lemma 5.25.** *For any  $k \in \mathbb{N}$ , we have a descent inequality as the following:*

$$V(\tilde{G}_k) - V(\tilde{G}_{k+1}) \geq \frac{2g_{k+1}^2}{cg_k^2} - n\lambda_{k+1}, \quad (169)$$

where  $g_k := \|F'(x_k)\|$ .

*Proof.* By the optimality condition in (92), we have  $\tilde{G}_k u_k + v_{k+1} = -\nabla f(x_k)$  and by definition of  $J_k$ , we have  $J_k u_k = \nabla f(x_{k+1}) - \nabla f(x_k)$ . Using Lemma 5.5, we have the following estimation:

$$\begin{aligned} V(\tilde{G}_k) - V(\tilde{G}_{k+1}) &= \nu(J_k, \tilde{G}_k, u_k) \geq \frac{\|\tilde{G}_k - J_k\| u_k\|^2}{u_k^\top \tilde{G}_k u_k} = \frac{\|\nabla f(x_{k+1})\|^2}{u_k^\top \tilde{G}_k u_k} \\ &\geq \frac{\|F'(x_{k+1})\|^2}{2(f(x_k) - f(x_{k+1}))} \geq \frac{2\|F'(x_{k+1})\|^2}{c\|F'(x_k)\|^2} = \frac{2g_{k+1}^2}{cg_k^2}, \end{aligned} \quad (170)$$

where the second inequality holds due to Lemma 5.23 and the third inequality holds due to (20). The rest of this proof remains the same as Theorem 5.22  $\square$

**Proof of Theorem 4.13.** For simplicity, we denote  $\gamma_k^2 := \frac{2g_{k+1}^2}{c^2 g_k^2}$ . Summing the result in Lemma 5.25 from  $k = 0$  to  $N - 1$ , we obtain

$$V(\tilde{G}_0) - V(\tilde{G}_N) \geq \sum_{k=0}^{N-1} \gamma_k^2 - n \sum_{k=1}^N \lambda_k. \quad (171)$$

Since, for every  $k$ , our method keeps  $\tilde{G}_k \succeq J_k$  and  $G_{k+1} \succeq J_k$  (see Lemma 5.15), we have  $V(\tilde{G}_k) > 0$  and therefore

$$V(\tilde{G}_0) \geq \sum_{k=0}^{N-1} \gamma_k^2 - n \sum_{k=1}^N \lambda_k. \quad (172)$$

By Lemma 5.24, we obtain

$$\left( V(G_0) + n\sqrt{L_H}C_f + nL_H C_r \right) \geq \sum_{k=k_0}^{N-1+k_0} \gamma_k^2. \quad (173)$$

We set  $C_{\text{GD}} := (V(G_0) + n\sqrt{L_H}C_f + nL_H C_r)$ . Dividing (173) by  $N$ , we obtain

$$\frac{C_{\text{GD}}}{N} \geq \frac{1}{N} \sum_{k=0}^{N-1} \gamma_k^2. \quad (174)$$

We derive from (174) with the concavity and monotonicity of  $\log x$  that

$$\log\left(\frac{C_{\text{GD}}}{N}\right) \geq \log\left(\frac{1}{N} \sum_{k=0}^{N-1} \gamma_k^2\right) \geq \frac{1}{N} \sum_{k=0}^{N-1} \log(\gamma_k^2) = \log\left(\left(\prod_{k=0}^{N-1} \frac{2g_{k+1}^2}{c^2 g_k^2}\right)^{1/N}\right) \quad (175)$$

Therefore, we have

$$g_N \leq \left(\frac{c^2 C_{\text{GD}}}{2N}\right)^{N/2} g_0. \quad (176)$$

□

## 6 Experiments

In the following section, we consider three applications from regression problems and image processing to provide numerical evidence about the superior performance of our algorithms on convex and nonconvex problems with KL inequality. While our algorithms are capable of solving non-smooth additive composite optimization problems, we focus solely on smooth problems, where our convergence analysis is also novel. This is mainly because efficiently solving sub-problems in the non-smooth case requires further investigations, possibly along the lines of [3], which we plan to address in future work. In this section, we compare our algorithms with various algorithms including the gradient descent method (GD), Nesterov accelerated gradient descent (NAG), the Heavy ball method (HB), cubic regularized Newton method (Cubic Newton) [42] and gradient regularized Newton method (Grad Newton) [41]. Here, we use the solver developed in [41] to solve the subproblems that include cubic terms.

### 6.1 Quadratic function

For this experiment, we test our methods: Grad SR1 PQN (Algorithm 2) on the convex quadratic problem with kernel:

$$\min_{x \in \mathbb{R}^n} f(x), \quad \text{with } f(x) = \frac{1}{2} \|Ax - b\|^2, \quad (177)$$

where  $A \in \mathbb{R}^{m \times n}$  with  $m < n$  has non-trivial kernel and  $b \in \mathbb{R}^m$ . This function  $f$  is convex but not strongly convex since the Hessian  $A^\top A \in \mathbb{R}^{n \times n}$  has 0 eigenvalue. The Lipschitz constant is  $L = \|A^\top A\|$  and  $L_H = 0$ . By calculation, we can show that the function  $f$  is gradient dominated, i.e., for any  $x \in \mathbb{R}^n$ ,

$$f(x) - \inf f \leq c \|\nabla f(x)\|^2,$$

for some  $c > 0$  (see [34]). If there exists  $y$  such that  $\nabla f(y) = A^\top (Ay - b) = 0$ , then  $f(y) = \inf f$  due to the convexity of  $f$ . Though this function is no longer coercive, the convergence guarantee remains valid with  $L_H = 0$  and  $\lambda_k \equiv 0$  for any  $k \in \mathbb{N}$ , as predicted by Theorem 4.2 or 4.9. In our experiment,  $A \in \mathbb{R}^{m \times n}$  and  $b \in \mathbb{R}^m$  are generated randomly with  $m = 250$  and  $n = 300$ . Since  $\lambda_k \equiv 0$  for all  $k \in \mathbb{N}$ , there is no distinction among Cubic-, Grad-, and classical SR1

methods. If the quasi-Newton metric  $G_k$  is invertible, we utilize the Sherman-Morrison formula for efficient implementation. As depicted in Figure 1, the SR1 method achieves a super-linear rate of convergence even on the quadratic problem with a non-trivial kernel. Here, the stepsize of NAG and GD is set to  $1/L$ .

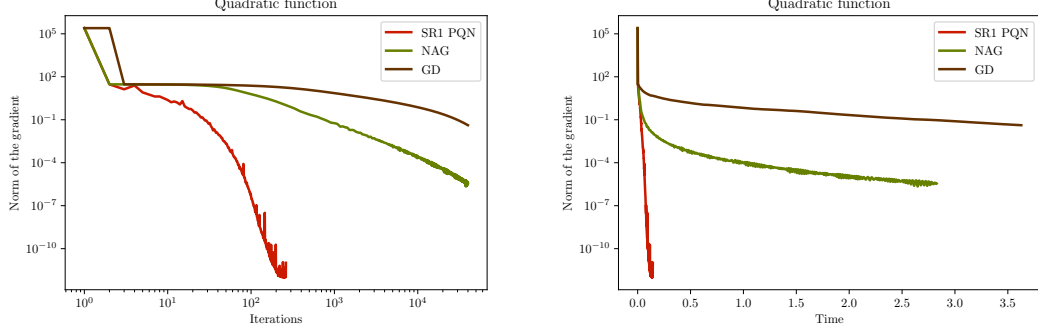


Figure 1: SR1 method significantly outperforms first-order methods in number of iterations and in terms of time.

### 6.1.1 Logistic regression with a convex regularization

The second experiment is a logistic regression problem with regularization on the benchmark dataset “mushroom” from UCI Machine Learning Repository [53]:

$$\min_{x \in \mathbb{R}^n} f(x), \quad \text{with } f(x) := \frac{1}{m} \sum_{i=1}^m \log(1 + \exp(-b_i a_i^\top x)) + \mu \sqrt{\|x\|^2 + \epsilon}, \quad (178)$$

where  $a_i \in \mathbb{R}^n$ ,  $b_i \in \mathbb{R}$  for  $i = 1, 2, \dots, m$  denote the given data from the “mushrooms” dataset. We set  $\epsilon = 1$ . The Lipschitz constants are  $L = 2 \sum_{i=1}^m \|a_i\|^2 + 2\mu$  and  $L_H = 4 \max_{i=1, \dots, m} \|a_i\|$ . However, in this experiment, we heuristically set  $L_H = 4$ . Here,  $m = 8124$  and  $n = 117$ . We set  $\bar{\kappa} = 2L$  for Cubic- and Grad SR1 PQN. The number of restarting steps for Grad SR1 PQN is 1 and the number for Cubic SR1 PQN is 14. We set the stepsize as  $1/L$  for NAG and GD. Note that  $f$  is coercive and strongly convex on any compact sublevel set. Thus, KL inequality holds with respect to  $\varphi(t) = ct^{1/2}$  for some  $c > 0$  and for any  $x \in \{x | f(x) \leq f(x_0)\}$ .

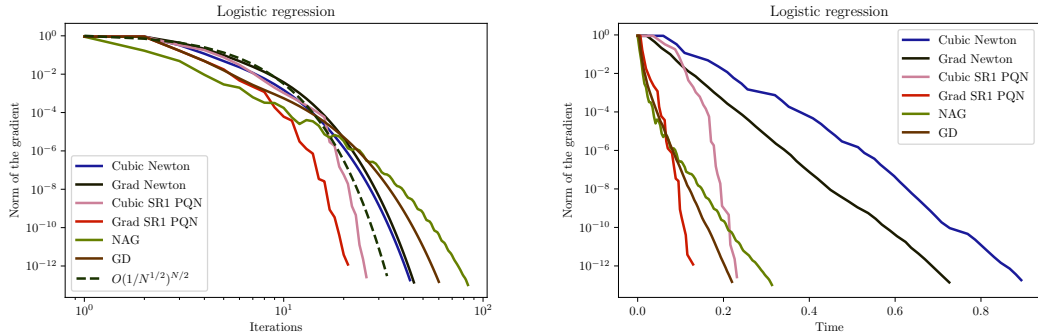


Figure 2: Cubic- and Grad SR1 PQN significantly outperform first-order methods in number of iterations, exhibiting super-linear convergence. In terms of time, Grad SR1 PQN remains one of the fastest methods due to the SR1 metric and the cheap computation of both the correction step and the restarting step.

As depicted in Figure 2, our two regularized quasi-Newton methods achieve a super-linear rate of convergence. However, due to the cubic term, neither Cubic Newton nor Cubic quasi-Newton is competitive when it comes to measuring the actual computation time. Grad SR1 PQN outperforms all other first-order methods significantly in number of iterations and is among the fastest in terms of time.

### 6.1.2 Image deblurring problem with a non-convex regularization

The third experiment focuses on an image deblurring problem with non-convex regularization. Let  $b \in \mathbb{R}^{MN}$  represent a blurry and noisy image, where each pixel  $b_i$  falls within the range  $[0, 1]$  for any  $i \in \{1 \dots MN\}$ . Here,  $b$  is constructed by stacking the  $M$  columns of length  $N$ . To recover a clean image  $x \in \mathbb{R}^{MN}$ , we solve the following optimization problem:

$$\min_{x \in \mathbb{R}^n} f(x), \quad \text{with } f(x) := \frac{1}{2} \|Ax - b\|^2 + \frac{\mu}{2} \log(\rho + \|Kx\|^2), \quad (179)$$

where the blurring operator  $A \in \mathbb{R}^{MN \times MN}$  is a linear mapping and  $Kx$  computes the forward differences in horizontal and vertical direction with  $\|K\| \leq 2\sqrt{2}$ . We set  $\mu = 0.001$  and  $\rho = 0.1$ . The Lipschitz constants are estimated from above as  $L = \|A\| + 5$  and  $L_H = 10$ . For this experiment, we take  $M = N = 64$ . Additionally, we set  $\bar{\kappa} = 2L$  for both Cubic- and Grad SR1 PQN. We set the quasi-Newton metric as  $\nabla^2 f(x_k)$  instead of  $LI$  for both the initialization and the restarting steps. In this case, our convergence analysis remains valid. The number of restarting steps for Cubic SR1 PQN is 13. For nonconvex optimization, we employ a line search with a backtracking strategy for both the gradient descent method (GD\_BT) and the Heavy ball method (HB\_BT), the latter being a special case of the iPiano algorithm from [44]). Figure 3 exhibits the comparison between our cubic SR1 method and GD\_BT and HB\_BT.

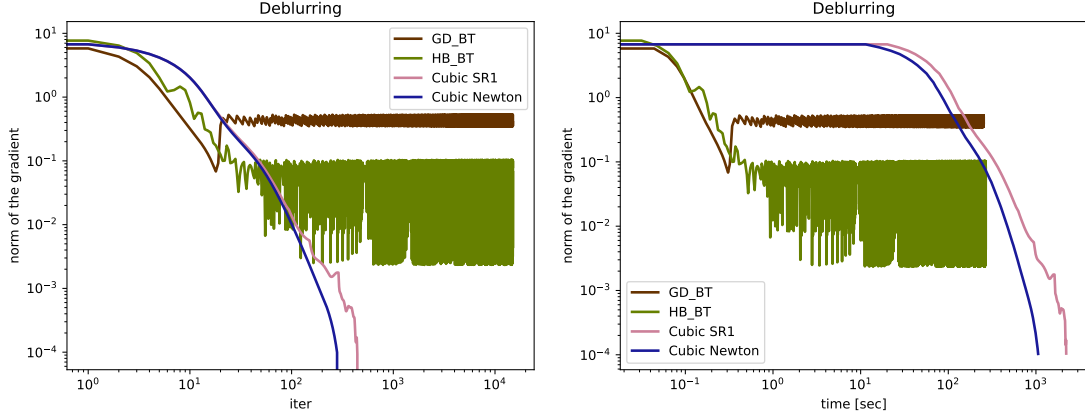


Figure 3: In the number of iterations and the norm of the gradients, Cubic Newton and Cubic SR1 PQN attain higher accuracy eventually. Moreover, both cubic Newton and Cubic SR1 PQN show more stable convergence, compared to first-order methods that utilize line search.

As illustrated in Figure 3, our Cubic SR1 PQN maintains superlinear convergence, despite slightly slower than the cubic Newton method. In the case when only the first-order information is accessible, Cubic SR1 PQN offers a more practical alternative. Both regularized second order methods are more stable than first-order method with Armijo line search when applied on this nonconvex problem. The oscillation in the latter methods may be attributed to the allowance of a large step size and a sharp local minimum of the nonconvex function.

## 7 Conclusion

In this paper, we study two regularized proximal quasi-Newton SR1 methods which converge with explicit (non-asymptotic) super-linear rates even for some classes of non-convex non-smooth optimization problems. The key is the adaptation of convergence analysis from the work [54] on strongly convex problems to non-convex problems that, instead, satisfies the KL property, illustrating the power of the trace potential function  $V(G) = \text{tr} G$ . Moreover, our algorithm and analysis is directly developed for non-smooth additive composite problems, although they are already novel in the smooth case. The analysis of convergence used in this paper, for the first time, reveals the possibility to study non-asymptotic super-linear convergence rates of classical quasi-Newton schemes without assuming strong convexity. This work shows that KL property, which can be observed in many non-convex problems, is an important generalization of the strong convexity to

study quasi-Newton methods and we can expect more results similar as in the strongly convex cases.

## A Convergence of cubic quasi-Newton SR1 method with Łojasiewicz inequality

**Assumption 3.**  *$f$  satisfies the global Łojasiewicz inequality with constant  $\mu$ , i.e., there exists a constant  $\mu$  such that for any  $x \in \mathbb{R}^n$ , we have*

$$f(x) - \min f \leq \frac{1}{2\mu} \|\nabla f(x)\|^2. \quad (180)$$

**Remark A.1.** *This assumption is equivalent to assume  $f$  has a quadratic growth rate for twice continuously differentiable  $f$  [46], i.e.,  $f(x) - \min f \geq c(\text{dist}(x, \text{argmin } f))^2$  for some constant  $c > 0$ .*

In order to solve the problem in (4) with  $g = 0$  and  $f$  satisfying (180), we employ the same cubic regularized SR1 quasi-Newton method (Algorithm 3). Algorithm 3 is the same with the first algorithm proposed in previous section where  $g = 0$ .

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### Algorithm 3 Cubic SR1 QN

---

**Require:**  $x_0, G_0 = LI, r_{-1} = 0$ .

**Update for**  $k = 0, \dots, N$ :

1. **Update**

$$x_{k+1} \in \underset{x \in \mathbb{R}^n}{\text{argmin}} \left\{ \langle \nabla f(x_k), x - x_k \rangle + \frac{1}{2} \|x - x_k\|_{(G_k + L_H r_{k-1} I)}^2 + \frac{L_H}{3} \|x - x_k\|^3 \right\}.$$

2. **Set**  $u_k = x_{k+1} - x_k, r_k = \|u_k\|, \lambda_k = L_H(r_{k-1} + r_k)$ .

3. **Compute**  $\tilde{G}_{k+1} = G_k + \lambda_k I$  and

$$F'(x_{k+1}) = \nabla f(x_{k+1}) - \nabla f(x_k) - \tilde{G}_{k+1} u_k.$$

4. **Compute**  $G_{k+1} = \text{SR1}(J_k, \tilde{G}_{k+1}, u_k)$  ( $J_k := \int_0^1 \nabla^2 f(x_k + tu_k) dt$ ).

5. **If**  $\|F'(x_{k+1})\| = 0$ : Terminate.

**End**

---

**Theorem A.2.** *Let Assumption 3 hold and  $g = 0$ . For any initialization  $x_0 \in \mathbb{R}^n$  and any  $N \in \mathbb{N}$ , Cubic SR1 QN has a global convergence with the rate:*

$$\|\nabla f(x_N)\| \leq \frac{6}{\mu} \left( \frac{C}{N^{1/3}} \right)^{N/2} \|\nabla f(x_0)\|,$$

where  $C = \left( 2nL + nR + 2nL_H \left( \frac{6}{L_H} (f(x_0) - \min f) \right)^{1/3} \right)$  and  $R = (6(f(x_0) - \min f)/L_H)^{1/3}$ .

### A.1 Convergence analysis of Algorithm 3

The optimality condition of the update step in (1) is

$$\nabla f(x_k) + (G_k + L_H r_{k-1} I)(x_{k+1} - x_k) + L_H \|x_{k+1} - x_k\| (x_{k+1} - x_k) = 0, \quad (181)$$

which, using our notations, simplifies to

$$\nabla f(x_k) + \underbrace{(G_k + \lambda_k I)}_{\tilde{G}_{k+1}} (x_{k+1} - x_k) = 0. \quad (182)$$

Surprisingly, using our notations,  $\tilde{G}_{k+1} = G_k + \lambda_k I$ .



**Lemma A.3.** For any  $k \in \mathbb{N}$ , we have

$$\tilde{G}_{k+1} \succeq G_{k+1} \succeq J_k \quad (183)$$

*Proof.* We argue by induction. When  $k = 0$ , we have  $G_0 = LI \succeq J_0$  and  $\tilde{G}_1 \succeq J_0$ . Then,  $\tilde{G}_1 \succeq G_1 \succeq J_0$ , thanks to Lemma 5.4. We assume the inequality holds for  $k$ , namely,  $\tilde{G}_k \succeq G_k \succeq J_{k-1}$ . Thus, for  $k + 1$ , we deduce that  $\tilde{G}_{k+1} = G_k + L_H(r_k + r_{k-1})I \succeq J_{k-1} + L_H(r_k + r_{k-1})I \succeq J_k$ . Thanks to Lemma 5.4, we obtain  $\tilde{G}_{k+1} \succeq G_{k+1} \succeq J_k$ . Therefore,  $\text{tr } \tilde{G}_{k+1} \geq \text{tr } G_{k+1}$ .  $\square$

**Lemma A.4.** For each  $k \in \mathbb{N}$ , we have

$$\begin{aligned} f(x_{k+1}) - f(x_k) &\leq -\frac{L_H}{6}r_k^3, \\ \langle (\tilde{G}_{k+1} - J_k)u_k, u_k \rangle &\leq \frac{6}{\mu} \|\nabla f(x_k)\|^2. \end{aligned}$$

*Proof.* We start with showing the first result. The update step 1 implies that

$$f(x_k) + \langle \nabla f(x_k), u_k \rangle + \frac{1}{2} \langle (G_k + L_H r_{k-1} I)u_k, u_k \rangle + \frac{L_H}{3}r_k^3 \leq f(x_k). \quad (184)$$

Then, thanks to Lemma 5.1, we have

$$f(x_{k+1}) - f(x_k) \leq \langle \nabla f(x_k), u_k \rangle + \frac{1}{2} \langle \nabla^2 f(x_k)u_k, u_k \rangle + \frac{L_H}{6} \|u_k\|^3. \quad (185)$$

Combining two inequalities above, we can obtain that

$$\begin{aligned} f(x_{k+1}) &\leq f(x_k) + \frac{1}{2} \langle \nabla^2 f(x_k)u_k, u_k \rangle - \frac{1}{2} \langle (G_k + L_H r_{k-1} I)u_k, u_k \rangle - \frac{L_H}{6} \|u_k\|^3 \\ &\leq f(x_k) - \frac{L_H}{6} r_k^3, \end{aligned} \quad (186)$$

where the last inequality holds thanks to the fact  $G_k + L_H r_{k-1} I \succeq J_{k-1} + \frac{L_H}{2} r_{k-1} I \succeq \nabla^2 f(x_k)$  (see Lemma 5.3). For the second result, we need another inequality from Lemma 5.1:

$$f(x_{k+1}) - f(x_k) \leq \langle \nabla f(x_k), u_k \rangle + \frac{1}{2} \langle J_k u_k, u_k \rangle + \frac{L_H}{6} \|u_k\|^3. \quad (187)$$

Adding (184) to (187), we obtain that

$$\langle (\tilde{G}_{k+1} - J_k)u_k, u_k \rangle \leq 2(f(x_k) - f(x_{k+1})) + \frac{2L_H}{3}r_k^3. \quad (188)$$

Combining the first result above, we can derive that

$$\begin{aligned} \langle (\tilde{G}_{k+1} - J_k)u_k, u_k \rangle &\leq 2(f(x_k) - f(x_{k+1})) + \frac{2L_H}{3}r_k^3 \\ &\leq 6(f(x_k) - f(x_{k+1})) \leq 6(f(x_k) - \inf f) \\ &\leq \frac{6}{\mu} \|\nabla f(x_k)\|^2, \end{aligned} \quad (189)$$

where the last inequality utilizes Łojasiewicz inequality.  $\square$

**Lemma A.5.** Given a number of iterations  $N \in \mathbb{N}$ , we have:

$$\sum_{k=0}^N r_k^3 \leq \frac{6(F(x_0) - \inf F)}{L_H} < +\infty, \quad (190)$$

$$\sum_{k=0}^{N-1} r_k \leq C_0 N^{\frac{2}{3}}, \quad (191)$$

$$\sum_{k=1}^N r_k \leq C_0 N^{\frac{2}{3}}, \quad (192)$$

where  $C_0 := \left(\frac{6(F(x_0) - \inf F)}{L_H}\right)^{1/3}$ .

*Proof.* The first inequality is derived from Lemma A.4. Then, we apply the Cauchy–Schwarz inequality and obtain

$$\sum_{i=0}^{N-1} r_k \leq \left( \sum_{k=0}^{N-1} r_k^3 \right)^{1/3} \left( \sum_{i=0}^{N-1} 1^{\frac{3}{2}} \right)^{2/3} \leq \left( \sum_{k=0}^{\infty} r_k^3 \right)^{1/3} N^{2/3} \leq C_0 N^{2/3}. \quad (193)$$

Similarly, we have

$$\sum_{i=1}^N r_k \leq \left( \sum_{k=1}^N r_k^3 \right)^{1/3} \left( \sum_{i=1}^N 1^{\frac{3}{2}} \right)^{2/3} \leq \left( \sum_{k=0}^{\infty} r_k^3 \right)^{1/3} N^{2/3} \leq C_0 N^{2/3}. \quad (194)$$

□

**Lemma A.6.**

$$V(\tilde{G}_{k+1}) - V(\tilde{G}_{k+2}) \geq \frac{\mu g_{k+1}^2}{6g_k^2} - n\lambda_{k+1}, \quad (195)$$

where  $g_k := \|\nabla f(x_k)\|$ .

*Proof.* Wlog, we assume  $(\tilde{G}_{k+1} - J_k)u \neq 0$ .

$$V(\tilde{G}_{k+1}) - V(G_{k+1}) = \nu(J_k, \tilde{G}_{k+1}, u_k) = \frac{u_k^\top (\tilde{G}_{k+1} - J_k)(\tilde{G}_{k+1} - J_k)u_k}{u_k^\top (\tilde{G}_{k+1} - J_k)u_k},$$

Using the definition of  $\tilde{G}_{k+1}$ , we have  $(\tilde{G}_{k+1} - J_k)u_k = \nabla f(x_k) - \nabla f(x_{k+1}) - \nabla f(x_k) = -\nabla f(x_{k+1})$ . Thanks to the second inequality from Lemma A.4, we obtain

$$\nu(J_k, \tilde{G}_{k+1}, u_k) \geq \frac{\mu \|\nabla f(x_{k+1})\|^2}{6 \|\nabla f(x_k)\|^2}.$$

On the other hand,

$$V(G_{k+1}) - V(\tilde{G}_{k+2}) = -n\lambda_{k+1}.$$

□

**Theorem A.7.** For any initialization  $x_0 \in \mathbb{R}^n$  and any  $N \in \mathbb{N}$ , Cubic SR1 QN has a global convergence with the rate:

$$\|\nabla f(x_N)\| \leq \frac{6}{\mu} \left( \frac{C}{N^{1/3}} \right)^{N/2} \|\nabla f(x_0)\|,$$

where  $C = \left( 2nL + 2nL_H R + 2nL_H \left( \frac{6}{L_H} (f(x_0) - \min f) \right)^{1/3} \right)$ . Here,  $R = \left( \frac{6(F(x_0) - \inf F)}{L_H} \right)^{1/3}$ .

*Proof of Theorem A.2.* Summing up the following inequalities from  $k = 0$  to  $k = N - 1$

$$V(\tilde{G}_{k+1}) - V(\tilde{G}_{k+2}) \geq \frac{\mu g_{k+1}^2}{6g_k^2} - n\lambda_{k+1},$$

we have

$$V(\tilde{G}_1) - V(\tilde{G}_N) + n \sum_{k=0}^{N-1} \lambda_k \geq \sum_{k=0}^{N-1} \frac{\mu g_{k+1}^2}{6g_k^2}.$$

Thanks to Lemma A.3,  $V(\tilde{G}_N) \geq V(J_{N-1}) \geq -nL$ ,  $V(\tilde{G}_1) \leq nL + 2nL_H R$ . According to Lemma A.5,  $\sum_{k=1}^N \lambda_k \leq 2L_H C_0(N^{2/3})$  and (A.5) implies that for any  $k \in \mathbb{N}$ ,  $r_k \leq R$  where  $R = \left( \frac{6(F(x_0) - \inf F)}{L_H} \right)^{1/3}$ . Dividing both sides by  $N$  and using the concavity of log, we have

$$\begin{aligned} \log \left( \frac{1}{N} \left( 2nL + 2nL_H R + 2L_H C_0(N^{2/3}) \right) \right) &\geq \log \left( \frac{1}{N} \left( \sum_{k=0}^N \frac{\mu g_{k+1}^2}{6g_k^2} \right) \right) \geq \left( \frac{1}{N} \sum_{k=0}^N \log \left( \frac{\mu g_{k+1}^2}{6g_k^2} \right) \right) \\ &\geq \log \left( \frac{\mu}{6} \left( \frac{g_N}{g_0} \right)^{2/N} \right). \end{aligned} \quad (196)$$

□

## B A summable sequence

**Lemma B.1** ([18]). *Let  $\rho \in (0, 1)$ ,  $(a_k)_{k \in \mathbb{N}}$  and  $(b_k)_{k \in \mathbb{N}}$  be two nonnegative sequences such that  $b_k$  is summable and  $a_{k+1} \leq (1 - \rho)a_k + b_k$ . Then  $a_k$  is summable. More precisely,  $\sum_{k=0}^{+\infty} a_k \leq \rho^{-1}(a_0 + \sum_{k=0}^{+\infty} b_k)$ .*

*Proof.* See [18, Lemma 3.1]. □

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